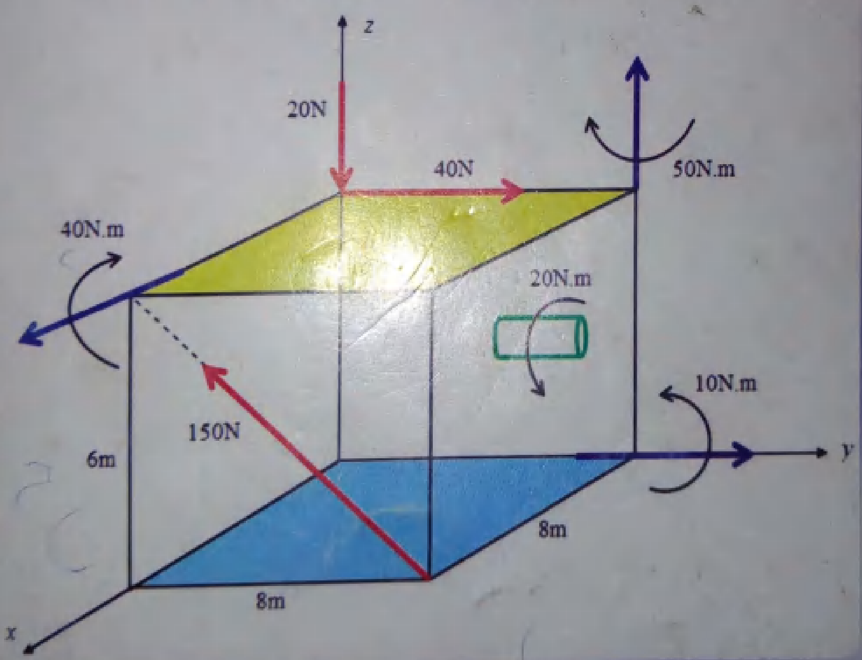
A HAND BOOK OF APPLIED MATHEMATICS-I WITH ENGINEERING MECHANICS-I

PROBLEM SOLVING APPROACH

For Engineering, Science and Technology Students

Force-Couple System and Wrench Resulatnt



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APPLIED MATHEMATICS-I

WITH

ENGINEERING MECHANICS-I

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By Begashaw Moltot

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"Talk to it you are with a teacher if you are with this book"

"The BOMBEA is working while he is eating"

The *ibid* proverb of early Chinese!

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CHAPTER-1

MATRICES AND DETERMINANTS

1.1 Definition of a Matrix

Definition: A rectangular array of numbers of the form $A=\begin{pmatrix} a_{11} & a_{12} \dots a_{1n} \\ a_{21} & a_{22} \dots a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \dots a_{mn} \end{pmatrix}$

where each $a_{ij} \in R$ is called a *Matrix* and the notation is abbreviated as $A=(a_{ij})_{m\times n}$. The horizontal arrangement in the array is called a row while the vertical arrangement is called column. Here, we say that A has size or order m by n written as $m\times n$. That means it has m rows and n columns. An element a_{ij}, $1\le i\le m$, $1\le j\le n$ is called the $(i,j)^{th}$ entry of A. It is an element in the i^{th} row and j^{th} column of A.

Example: Let $A=\begin{pmatrix} 2 & 3 & 4 \\ -3 & -4 & 5 \end{pmatrix}$. Here, A has 2 rows and 3 columns and thus the size of A is 2×3 and $a_{11}=2, a_{12}=3, a_{13}=4, a_{21}=-3, a_{22}=-4, a_{23}=5$.

Equality of Matrices: Two matrices A and B are said to be equal (written as $A=B$) if and only if A and B have the same size and all their corresponding entries are equal.

That is, if $A=(a_{ij})_{m\times n}, B=(b_{ij})_{m\times n}$ with $a_{ij}=b_{ij}$ for all $1\le i\le m,\ 1\le j\le n$.

Examples:

1. Let $A=\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}, B=\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}, C=\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$

Here, $A=B$ but $A\ne C$. (Why?)

2. Find m, n and p such that $\begin{pmatrix} m^2+2 & 3 \\ n-3 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 2p-3 \\ 1 & 4 \end{pmatrix}$.

Solution: Using equality of matrices, we have

$$\begin{pmatrix} m^2+2 & 3 \\ n-3 & 4 \end{pmatrix} = \begin{pmatrix} 6 & 2p-3 \\ 1 & 4 \end{pmatrix} \Rightarrow \begin{cases} m^2+2=6 \\ n-3=1 \\ 2p-3=3 \end{cases} \Rightarrow \begin{cases} m^2=4 \\ n=4 \\ 2p=6 \end{cases} \Rightarrow m=\pm 2, n=4, p=3$$

1.2 Types of Matrices

a) ***The zero matrix***: If all the entries of a matrix are zero, then the matrix is called **zero matrix** or **null matrix** and denoted by the symbol $\mathbf{O} = (0_{ij})_{m\times n}$.

For instance, $\mathbf{O} = (0 \quad 0), \mathbf{O} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are zero matrices.

b) ***Row matrix:*** A matrix having exactly one row is called *row matrix* and denoted by $A = (a_{1j})_{1\times n}$. For instance, $A = (2 \quad 3)$ and $B = (2 \quad -3 \quad 4)$ are row matrices but $C = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ is not a row matrix.

c) ***Column Matrix:*** A matrix having exactly one column is called *column matrix* and denoted by $A = (a_{i1})_{m\times 1}$. For instance, the matrix $A = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a column matrix but $B = (2 \quad -3)$ is not a column matrix.

d) ***Square matrix***: A matrix in which the numbers of rows and the number of columns are equal, that is, an $n \times n$ matrix, is called a *square matrix* of order n. In a square matrix, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called *diagonal entries*. The diagonal containing each entry $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ is called the *main diagonal* or *principal diagonal* of A. The entries that are not in the main diagonal are called *non-diagonal* (*off-diagonal*) entries.

The sum of the diagonal entries of a square matrix A is called *trace* of A, denoted by $tr(A)$. That is, $tr(A) = a_{11} + a_{22} + a_{33} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$.

Example: The matrix $A = \begin{pmatrix} 2 & -3 & 4 \\ 3 & -4 & 5 \\ 4 & 5 & 6 \end{pmatrix}$ is a square matrix of order 3 with diagonal entries $a_{11} = 2$, $a_{22} = -4, a_{33} = 6$ and $tr(A) = a_{11} + a_{22} + a_{33} = 4$.

e) ***Diagonal Matrix***: A square matrix $A = (a_{ij})_{n\times n}$ is called a *diagonal matrix* if and only if all of its non-diagonals entries are zero.

For instance, $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ are diagonal matrices.

f) ***Scalar matrix:*** A diagonal matrix in which all the diagonal entries are equal is called *Scalar matrix*.

For instance, the matrix $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is a scalar matrix.

g) ***Identity matrix:*** A scalar matrix with diagonal entry of 1 is called *identity matrix* denoted by I_n. For instance, the matrices $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is identity matrix.

h) ***Triangular matrix:*** Let A be a square matrix. If all entries of A above the main diagonal are zero, A is called *lower* triangular and if all entries of A below the main diagonal are zero, A is called *upper* triangular. Thus, a *triangular* matrix is a matrix which is either lower or upper triangular. For instance, $A = \begin{pmatrix} 2 & 0 & 0 \\ 3 & -4 & 0 \\ 4 & 5 & 6 \end{pmatrix}$ is lower triangular and $B = \begin{pmatrix} 2 & -3 & 4 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$ is upper triangular. Hence, A and B are triangular matrices.

1.3 Operations on Matrices

Let $A=(a_{ij})_{m\times n}$ and $B=(b_{ij})_{m\times n}$ be any two matrices. Then,

i) **Addition of matrices:**
The sum $A+B$ is obtained by adding corresponding entries as $A+B=(a_{ij}+b_{ij})_{m\times n}$

ii) **Subtraction:** The difference $A-B$ is obtained by subtracting corresponding entries as $A-B=(a_{ij}-b_{ij})_{m\times n}$.

Remark: Two matrices can be added or subtracted if they have the same size.

iii) **Scalar multiplication:** The scalar multiple kA of A is a matrix obtained by multiplying each entries of A by the scalar k as $kA=(ka_{ij})_{m\times n}$.

Particularly, consider 2×2 matrices $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

Addition: $A+B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}$

Subtraction: $A-B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} a_{11}-b_{11} & a_{12}-b_{12} \\ a_{21}-b_{21} & a_{22}-b_{22} \end{pmatrix}$

Examples:

1. Let $A=\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}, B=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Then,

a) $A+B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}+\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}+\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}=\begin{pmatrix} 3+1 & 4+2 \\ 5+3 & 6+4 \end{pmatrix}=\begin{pmatrix} 4 & 6 \\ 8 & 10 \end{pmatrix}$

b) $A-B=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}-\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}=\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}-\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}=\begin{pmatrix} 3-1 & 4-2 \\ 5-3 & 6-4 \end{pmatrix}=\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

c) $3A=3\begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}=\begin{pmatrix} 3\times 3 & 3\times 4 \\ 3\times 5 & 3\times 6 \end{pmatrix}=\begin{pmatrix} 9 & 12 \\ 15 & 18 \end{pmatrix}$

2. Let $A=\begin{pmatrix}2 & 1\\ 3 & 3\end{pmatrix}$, $B=\begin{pmatrix}6 & -2\\ -4 & 5\end{pmatrix}$. Then, find C such that $A+B-C$ is zero matrix.

Solution: $A+B-C$ is zero matrix means

$$A+B-C=\mathbf{0}\Rightarrow C=A+B=\begin{pmatrix}2 & 1\\ 3 & 3\end{pmatrix}+\begin{pmatrix}6 & -2\\ -4 & 5\end{pmatrix}=\begin{pmatrix}8 & -1\\ -1 & 8\end{pmatrix}$$

3. Let $A=\begin{pmatrix}a-1 & 1\\ b & -4\end{pmatrix}$, $B=\begin{pmatrix}3 & 1\\ 1 & -2\end{pmatrix}$, $C=\begin{pmatrix}3 & c\\ 1 & -2\end{pmatrix}$. Find a,b and c such that $2A=3B+C$.

Solution: Using equality of matrices, we have

$$2A=3B+C\Rightarrow 2\begin{pmatrix}a-1 & 1\\ b & -4\end{pmatrix}=3\begin{pmatrix}3 & 1\\ 1 & -2\end{pmatrix}+\begin{pmatrix}3 & c\\ 1 & -2\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}2a-2 & 2\\ 2b & -8\end{pmatrix}=\begin{pmatrix}12 & c+3\\ 4 & -8\end{pmatrix}\Rightarrow\begin{cases}2a-2=12\\ 2b=4\\ c+3=2\end{cases}\Rightarrow\begin{cases}a=7\\ b=2\\ c=-1\end{cases}$$

4. Find x and y from the equation $\begin{pmatrix}2x & -3\\ -4 & 5x\end{pmatrix}+\begin{pmatrix}3y & 4\\ 5 & -3y\end{pmatrix}=\begin{pmatrix}7 & 1\\ 1 & 7\end{pmatrix}$.

Solution: Add the matrices and use equality of matrices.

$$\begin{pmatrix}2x & -3\\ -4 & 5x\end{pmatrix}+\begin{pmatrix}3y & 4\\ 5 & -3y\end{pmatrix}=\begin{pmatrix}7 & 1\\ 1 & 7\end{pmatrix}\Rightarrow\begin{pmatrix}2x+3y & 1\\ 1 & 5x-3y\end{pmatrix}=\begin{pmatrix}7 & 1\\ 1 & 7\end{pmatrix}$$

$$\Rightarrow\begin{cases}2x+3y=7\\ 5x-3y=7\end{cases}\Rightarrow 7x=14\Rightarrow x=2, y=1$$

5. Let $A=\begin{pmatrix}8 & -6 & 0\\ 0 & -4 & -9\\ 9 & 6 & 5\end{pmatrix}$. Find matrix B so that $2A-3B$ is identity matrix.

Solution: Since $2A-3B$ is identity, we have $2A-3B=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$.

Then, $-3B=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}-\begin{pmatrix}16 & -12 & 0\\ 0 & -8 & -18\\ 18 & 12 & 10\end{pmatrix}\Rightarrow B=\begin{pmatrix}5 & -4 & 0\\ 0 & -3 & -6\\ 6 & 4 & 3\end{pmatrix}$

6. If $A-B=\begin{pmatrix}1 & 6\\ 2 & -4\\ 0 & -5\end{pmatrix}$, $A+B=\begin{pmatrix}-5 & 0\\ 8 & 0\\ 4 & -7\end{pmatrix}$, find the matrices A and B.

Solution: Here, if we add the two equations, we get

$$2A=\begin{pmatrix}-4 & 6\\ 10 & -4\\ 4 & -12\end{pmatrix}\Rightarrow A=\frac{1}{2}\begin{pmatrix}-4 & 6\\ 10 & -4\\ 4 & -12\end{pmatrix}=\begin{pmatrix}-2 & 3\\ 5 & -2\\ 2 & -6\end{pmatrix}.$$

Again, $A+B=\begin{pmatrix}-5 & 0\\ 8 & 0\\ 4 & -7\end{pmatrix}\Rightarrow B=\begin{pmatrix}-5 & 0\\ 8 & 0\\ 4 & -7\end{pmatrix}-A\Rightarrow B=\begin{pmatrix}-3 & -3\\ 3 & 2\\ 2 & -1\end{pmatrix}$

7. Let $X=\begin{pmatrix}4 & 3 & -9\\ 0 & -2 & 15\\ -12 & 9 & -5\end{pmatrix}$. Find Y so that $2X+3Y$ is a scalar matrix whose non-zero entry is 2.

Solution: Given $2X+3Y$ is a scalar matrix with non-zero entry of 2. But a scalar matrix D with a non zero entry of a is given by the form $D=\begin{pmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & a\end{pmatrix}$.

In our case, since $a=2$ is given, we have $2X+3Y=D=\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$.

$$\text{Thus, } 2X+3Y=\begin{pmatrix}8 & 6 & -18\\ 0 & -4 & 30\\ -24 & 18 & -10\end{pmatrix}+3Y=\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}\Rightarrow Y=\begin{pmatrix}-2 & -2 & 6\\ 0 & 2 & -10\\ 8 & -6 & 4\end{pmatrix}$$

iv) Matrix Multiplication:

Let A and B be two matrices. We need to see how to find the product $A.B$. Let $R_1, R_2, R_3, \ldots, R_m$ be the rows of A and $C_1, C_2, C_3, \ldots, C_n$ be columns of B.

Then, the product of A and B is defined as follow:

Let the product be denoted by $P = \mathbf{AB} = (p_{ij})_{m\times n}$.

First: Compute the entry of the product using $p_{ij} = R_i C_j$. The operation in $R_i C_j$ is simply dot product.

Second: Put all the results in the general form as follow:

$$\text{The product is } P = (p_{ij})_{m\times n} = \mathbf{AB} = \begin{pmatrix} R_1C_1 & R_1C_2 & R_1C_3 & \cdots & R_1C_n \\ R_2C_1 & R_2C_2 & R_2C_3 & \cdots & R_2C_n \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ R_mC_1 & R_mC_2 & R_mC_3 & \cdots & R_mC_n \end{pmatrix}$$

Illustration: To make the idea of the above formula, more simple, particularly, consider 2×2 matrices $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$.

First: Identify rows of A and columns of B:

Rows of A: We have two rows: $R_1 = (a_{11} \quad a_{12})$, $R_2 = (a_{21} \quad a_{22})$

Columns of B: We have two columns: $C_1 = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$, $C_2 = \begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix}$

Second: Apply the above general formula to get the product. Taking dot product of rows of A with the corresponding columns of B, we have

$$\begin{cases} R_1C_1 = (a_{11} \quad a_{12})\begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = a_{11}.b_{11} + a_{12}.b_{21}, \\ R_1C_2 = (a_{11} \quad a_{12})\begin{pmatrix} b_{12} \\ b_{22} \end{pmatrix} = a_{11}.b_{12} + a_{12}.b_{22}. \\ R_2C_1 = a_{21}.b_{11} + a_{22}.b_{21}, \\ R_2C_2 = a_{21}.b_{12} + a_{22}.b_{22} \end{cases}$$

Therefore, by putting in the respective positions, the product is easily determine to be: $\mathbf{AB} = \begin{pmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{pmatrix} = \begin{pmatrix} a_{11}.b_{11} + a_{12}.b_{21} & a_{11}.b_{12} + a_{12}.b_{22} \\ a_{21}.b_{11} + a_{22}.b_{21} & a_{21}.b_{12} + a_{22}.b_{22} \end{pmatrix}$

Remarks: Compatibility Condition for Multiplication

The product AB is defined if and only if the number of columns of A is equal to the number of rows of B. Besides, if A is $m \times p$ and B is $p \times n$, then the product AB will be $m \times n$ matrix.

Examples:

1. Let $A = \begin{pmatrix} 1 & 2 \\ 0 & -3 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 3 \\ 1 & -1 \end{pmatrix}$. Then, find the product AB.

Solution: Since A and B are 2×2, the product is also 2×2.

So, the product has the form: $P = \mathbf{AB} = \begin{pmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{pmatrix}$

First: Identify rows of A and columns of B.

Rows of A: $R_1 = (1 \quad 2), R_2 = (0 \quad -3)$ and columns of B: $C_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, C_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

Second: Apply the above general formula to get the product.

That is $\begin{cases} R_1C_1 = (1 \quad 2)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2+2=0, \; R_1C_2 = (1 \quad 2)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = 3-2=1 \\ R_2C_1 = (0 \quad -3)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0-3=-3, \; R_2C_2 = (0 \quad -3)\begin{pmatrix} 3 \\ -1 \end{pmatrix} = 0+3=3 \end{cases}$

Hence, the product is $\mathbf{AB} = \begin{pmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & 3 \end{pmatrix}$.

2. If $A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 1 & -2 \\ 0 & 1 \end{pmatrix}$, then find the product AB.

Solution: $\mathbf{AB} = \begin{pmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 4 & -4 \end{pmatrix}$.

3. Given matrix A with size 3×4, matrix B with size 4×6 and matrix C with size 6×3. Then, which multiplication is possible? AB, AC, BC?

Solution: The products AB and BC are possible because for both number of columns of the first matrix is equal to the number of rows of the second matrix. But the product AC is not possible because the number of columns of the first matrix A is not equal to the number of rows of the second matrix C.

4. Let $A=\begin{pmatrix}1\\0\\2\end{pmatrix}$, $B=\begin{pmatrix}-2 & 3 & 1\end{pmatrix}$. Find the product **AB**.

Solution: The product is $\mathbf{AB}=\begin{pmatrix}R_1C_1 & R_1C_2 & R_1C_3\\ R_2C_1 & R_2C_2 & R_2C_3\\ R_3C_1 & R_3C_2 & R_3C_3\end{pmatrix}=\begin{pmatrix}-2 & 3 & 1\\ 0 & 0 & 0\\ -4 & 6 & 2\end{pmatrix}$

5. Suppose the size of A is $(2m-4)\times m$ and that of B is $(n-2)\times(n+1)$. Find the values of m and n such that the products AB and BA exist.

Solution: Use compatibility condition for multiplication. That is for the product of two matrices to be defined, always the number of columns of the first matrix must be equal to number of rows of the second matrix. For the product $A.B$ to be defined, the number of columns of A must be equal to the number of rows of B. That is $m=n-2$. For the product $B.A$ to be defined, the number of columns of B must be equal to the number of rows of A. That is $n+1=2m-4$.

Then, $\begin{cases}m=n-2\\ n+1=2m-4\end{cases}\Rightarrow n+1=2(n-2)-4\Rightarrow n=2n-8\Rightarrow n=9, m=7$

6. Find the values of the constants a, b, c, d if

a) $\begin{pmatrix}2 & 1\\ a & b\end{pmatrix}\begin{pmatrix}8 & -3\\ 0 & -2\end{pmatrix}=\begin{pmatrix}4c & d\\ -16 & 12\end{pmatrix}$ b) $\begin{pmatrix}a & 3\\ 2 & 6\end{pmatrix}\begin{pmatrix}4 & 7\\ -3 & b\end{pmatrix}=\begin{pmatrix}15 & 39\\ 5c & 2d\end{pmatrix}$

Solution: In each case, first find the products and then apply equality.

a) $\begin{pmatrix}2 & 1\\ a & b\end{pmatrix}\begin{pmatrix}8 & -3\\ 0 & -2\end{pmatrix}=\begin{pmatrix}4c & d\\ -16 & 12\end{pmatrix}\Rightarrow\begin{pmatrix}16 & -8\\ 8a & -3a-2b\end{pmatrix}=\begin{pmatrix}4c & d\\ -16 & 12\end{pmatrix}$

$\Rightarrow a=-2, b=-3, c=4, d=-8$

b) Similarly, we get $a=6, b=-1, c=-2, d=4$

Power of a matrix: The power of a square matrix A is the n^{th} power of A which is defined inductively as $A^0 = I, A^1 = A, A^2 = A.A, A^3 = A^2.A, ..., A^n = A^{n-1}.A$.

Examples:

1. Find n^{th} powers of A, B, C where $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Solution:

a) $A^2 = A.A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ (Observe that $c_{12} = 4 = 2.2$)

$A^3 = A^2.A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$ (Observe that $c_{12} = 6 = 2.3$)

Hence, $A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$

b) $B^2 = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 6 & 0 \end{pmatrix}$ (Observe that $c_{11} = 4 = 2^2$, $c_{21} = 6 = 3.2$)

$B^3 = B^2B = \begin{pmatrix} 4 & 0 \\ 6 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 8 & 0 \\ 12 & 0 \end{pmatrix}$ (Observe that $c_{11} = 8 = 2^3$, $c_{21} = 12 = 3.2^2$)

Hence, $B^n = \begin{pmatrix} 2^n & 0 \\ 3.2^{n-1} & 0 \end{pmatrix}$ for $n \geq 1$

c) $C^2 = C.C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; C^3 = C^2.C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So, $C^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for $n \geq 3$

2. Let $p(x) = 2x^2 - x - 7$ and $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$. Then, find $p(A)$.

Solution: $p(A) = 2A^2 - A - 7I = 2\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}^2 - \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} - 7\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 7 \\ 14 & -2 \end{pmatrix}$

1.4 Transpose of a matrix and Symmetric Matrices

Transpose and its properties:

A matrix obtained from a given matrix A by writing rows as columns or columns as rows is called **transpose** of A and denoted by A'.

Symmetric and Skew-Symmetric matrices:

A square matrix A is said to be

a) *Symmetric* if $A' = A$. That is $a_{ij} = a_{ji}$ for each i and j.

b) *Skew - symmetric* if $A' = -A$. That is $a_{ij} = -a_{ji}$ for each i and j.

Properties of Transpose:

Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$ and k be a scalar. Then,

a) $(A+B)' = A' + B'$ b) $(A')' = A$

c) $(A-B)' = A' - B'$ d) $(kA)' = kA'$

e) $(AB)' = B'A'$ (whenever AB is defined)

Remark: Decomposition of Formula

Any square matrix A can be expressed uniquely as a **sum** of symmetric and skew- symmetric matrices.

That is $A = S + K$ where S is symmetric and K is skew symmetric.

Question: How to find the symmetric part S and the skew symmetric part K?

For any square matrix A, take $S = \frac{1}{2}(A + A')$ and $K = \frac{1}{2}(A - A')$.

Here, $\begin{cases} S' = \left(\frac{1}{2}(A+A')\right)' = \frac{1}{2}(A' + (A')') = \frac{1}{2}(A + A') = S \\ K' = \left(\frac{1}{2}(A-A')\right)' = \frac{1}{2}(A' - (A')') = \frac{1}{2}(A' - A) = -\frac{1}{2}(A - A') = -K \end{cases}$

Besides, $S + K = \frac{1}{2}(A + A') + \frac{1}{2}(A - A') = \frac{1}{2}[A + A + A' - A'] = \frac{1}{2}(2A) = A$.

Therefore, $A = S + K = \underbrace{\frac{1}{2}(A + A')}_{\text{Symmetric part}} + \underbrace{\frac{1}{2}(A - A')}_{\text{Skew-symmetric part}}$

Examples:

1. Let $A=\begin{pmatrix}1&2&3\\2&3&4\\3&4&5\end{pmatrix}$, $B=\begin{pmatrix}0&1&2\\-1&0&3\\-2&-3&0\end{pmatrix}$, $C=\begin{pmatrix}3&2&1\\4&5&6\\7&8&9\end{pmatrix}$

a) Find A', B', C' and verify that A is symmetric, B is skew and C is neither.

b) Express matrix C as a sum of symmetric and skew symmetric matrices.

Solution: Here, use the definition of transpose as follow:

a) $A'=\begin{pmatrix}1&2&3\\2&3&4\\3&4&5\end{pmatrix}$, $B'=\begin{pmatrix}0&-1&-2\\1&0&-3\\2&3&0\end{pmatrix}$, $C'=\begin{pmatrix}3&4&7\\2&5&8\\1&6&9\end{pmatrix}$

Compare the given matrices and their transposes: Since $A'=A$, matrix A is symmetric. On the other hand, since $B'=-B$, matrix B is skew-symmetric. But $C'\neq C$ and $C'\neq -C$. So, C is neither symmetric nor skew-symmetric.

b) We need to express C as $C=S+K$ where S is symmetric and K is skew.

By the above decomposition:

$$S=\frac{1}{2}(C+C')=\frac{1}{2}[\begin{pmatrix}3&2&1\\4&5&6\\7&8&9\end{pmatrix}+\begin{pmatrix}3&4&7\\2&5&8\\1&6&9\end{pmatrix}]=\underbrace{\begin{pmatrix}3&3&4\\3&5&7\\4&7&9\end{pmatrix}}_{\text{Symmetric part}}$$

$$K=\frac{1}{2}(C-C')=\frac{1}{2}[\begin{pmatrix}3&2&1\\4&5&6\\7&8&9\end{pmatrix}-\begin{pmatrix}3&4&7\\2&5&8\\1&6&9\end{pmatrix}]=\underbrace{\begin{pmatrix}0&-1&-3\\1&0&-1\\3&1&0\end{pmatrix}}_{\text{Skew-symmetric part}}$$

2. Find a,b,c if $\begin{pmatrix}0&a^2+2&7\\-11&0&2b+1\\1-c&9&0\end{pmatrix}$ is skew-symmetric.

Solution: For A to be skew, $A'=-A\Rightarrow\begin{cases}a^2+2=11\\1-c=-7\\2b+1=-9\end{cases}\Rightarrow a=\pm 3, b=-5, c=8$

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3. Let $\begin{pmatrix} x & 6 & y \\ 9x & -3 & -6 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & -3x \\ -2 & 1 \\ 1-x & 2 \end{pmatrix}$. Find x, y if $A+3B'=0$.

Solution:

$$A+3B'=0 \Rightarrow \begin{pmatrix} x & 6 & y \\ 9x & -3 & -6 \end{pmatrix}+3\begin{pmatrix} 1 & -2 & 1-x \\ -3x & 1 & 2 \end{pmatrix}=0$$

$$\Rightarrow \begin{pmatrix} x+3 & 0 & y+3-3x \\ 0 & 0 & 0 \end{pmatrix}=0 \Rightarrow \begin{cases} x+3=0 \\ y+3-3x=0 \end{cases} \Rightarrow x=-3, y=12.$$

4. Let $A=\begin{pmatrix} -1 & 1 & 3 \\ 1 & 0 & 5 \end{pmatrix}$, $B=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 3 & 1 & 1 \end{pmatrix}$, $C=\begin{pmatrix} 2 & 5 \\ -2 & 3 \\ 4 & 1 \end{pmatrix}$, $D=\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$.

Compute (if possible) $2A'+3C$, AD, $5AC-D+(I_2)^3$

Solution: $2A'+3C=2\begin{pmatrix} -1 & 1 \\ 1 & 0 \\ 3 & 5 \end{pmatrix}+3\begin{pmatrix} 2 & 5 \\ -2 & 3 \\ 4 & 1 \end{pmatrix}=\begin{pmatrix} 4 & 17 \\ -4 & 9 \\ 18 & 13 \end{pmatrix}$

5. Let $A=\begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -4 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 2 & 4 \end{pmatrix}$. Find C if $C-2A=(3B)'$.

Solution: From property of transpose, for any constant k, $(kB)'=kB'$.

$$C-2A=(3B)' \Rightarrow C=2A+3B'=2\begin{pmatrix} 2 & -1 & -3 \\ -1 & 2 & -4 \end{pmatrix}+3\begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} 4 & -2 & -6 \\ -2 & 4 & -8 \end{pmatrix}+\begin{pmatrix} 3 & 9 & 6 \\ 6 & 0 & 12 \end{pmatrix}=\begin{pmatrix} 7 & 7 & 0 \\ 4 & 4 & 4 \end{pmatrix}$$

6. Given $A=\begin{pmatrix} 7 & x \\ 4 & 3 \\ 6 & -3 \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 1 & 3 \\ 2 & 4 & 4 \\ 5 & 7 & 6 \end{pmatrix}$. In $C=A'B$ if $c_{23}=-21$, find x.

Solution: $c_{23}=R_2(\text{of A}')\times C_3(\text{of B}) \Rightarrow 3x+12-18=-21 \Rightarrow x=-5$

1.5 Row Operations, Row –Echelon form and Rank of a matrix

Any of the following three actions performed on matrices is called *elementary row operation.*

i) Interchanging of two rows $R_i \leftrightarrow R_j$ (This means take R_i to R_j and R_j to R_i)

ii) Replacing a row by its non – zero scalar multiple $R_i \to kR_i$

iii) Adding a multiple of one row onto another row $R_j \to R_j + kR_i$

Definitions:

a) **Zero row:** A row of a matrix is said to be a *zero row* if all its entries are zero but if there is at least one non-zero entry, it is called *non-zero row.*

b) **Leading entry:** The leftmost nonzero entry (the first non-zero entry ingoing from left to right) of a non-zero row is called *leading entry.*

For instance, for $\begin{pmatrix} 0 & 2 & 0 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 6 \end{pmatrix}$, the first and the third rows are non-zero rows but the second row is zero row. So, the leading entry is 2 for the first row and 7 for the third row but the second row has no leading entry.

c) **Reduced-Row-echelon form (RREF):** A matrix is said to be in *reduced-row-echelon form* if it satisfies the following conditions:

i) All zero rows (if any) are below all the non-zero rows.

ii) For non – zero rows, the row with more zeros is below the row with less zeros (In each row, number of zeros are counted from left to right and no two non-zero rows have the same number of zeros).

iii) The leading entry (the left most non – zero entry) in each non– zero row is 1.

iv) All other elements of the column in which the leading entry 1 occurs (below and above the leading entry1) are zero.

d) **Row-echelon form (REF):** A matrix which satisfies only the first two conditions in the above definition is said to be in *row – echelon* form.

e) **Rank of a matrix:** Let A be an mxn arbitrary matrix. The number of non-zero rows in the reduced-row-echelon form (in the row-echelon form) of A is known as *rank of* A and denoted by *rank* (A).

Examples:

1. Justify whether the following matrices are row-echelon form, reduced row-echelon form or neither and give their ranks.

$$A=\begin{pmatrix}1&0&2&3&0\\0&1&1&1&0\\0&0&0&0&1\end{pmatrix}, B=\begin{pmatrix}0&0&0&1&2\\0&0&1&0&0\\0&0&0&1&0\end{pmatrix}, C=\begin{pmatrix}1&0&0&0&0\\0&0&0&0&1\\0&0&0&0&0\end{pmatrix}$$

Solution: Here, the matrices A and C are in reduced row-echelon form. So, their ranks are read to be $rank(A)=3$, $rank(C)=2$. But matrix B is not in reduced-row echelon form because the first and the third rows have the same number of zeros (counting from left to right).

2. Transform into reduced row - echelon form and determine their ranks.

$$a)\ B=\begin{pmatrix}1&-2\\4&5\\7&-1\\3&-6\end{pmatrix} \qquad b)\ C=\begin{pmatrix}0&3&0\\-2&0&4\\3&0&-6\\0&-5&0\\1&0&-2\end{pmatrix} \qquad c)\ A=\begin{pmatrix}1&2&5\\2&3&8\\-1&1&2\end{pmatrix}$$

Solution: In the first example, we saw how to identify whether the matrices are in reduced row echelon form or not by comparing with the definitions. But the basic target is how to transform a given matrix into reduced-row echelon form by applying the basic elementary row operations.

$$a)\ \begin{pmatrix}1&-2\\4&5\\7&-1\\3&-6\end{pmatrix}\xrightarrow[R_4\to R_4-3R_1]{R_3\to R_3-R_2}\begin{pmatrix}1&-2\\4&5\\3&-6\\0&0\end{pmatrix}\xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-4R_1}\begin{pmatrix}1&-2\\0&13\\0&0\\0&0\end{pmatrix}$$

$$\xrightarrow{R_2\to\frac{1}{13}R_2}\begin{pmatrix}1&-2\\0&1\\0&0\\0&0\end{pmatrix}\xrightarrow{R_1\to R_1+2R_2}\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}=rref(B)$$

As we can see the reduced row-echelon form of B, there are only two non-zero rows. Hence, rank of matrix B is $rank(B)=2$.

b) $\begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 4 \\ 3 & 0 & -6 \\ 0 & -5 & 0 \\ 1 & 0 & -2 \end{pmatrix} \xrightarrow[R_5 \to R_5 + \frac{1}{2}R_2]{R_3 \to R_3 + \frac{3}{2}R_2,\ R_4 \to R_4 + \frac{5}{3}R_1} \begin{pmatrix} 0 & 3 & 0 \\ -2 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} -2 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\xrightarrow[R_2 \to \frac{1}{3}R_2]{R_1 \to -\frac{1}{2}R_1} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \text{rref}(C).$$

Since only the first and the second rows of rref(C) are non-zero, rank = 2.

Note: Every matrix can be changed into RREF. Besides, the RREF of every matrix is unique. That means every matrix has exactly one RREF but different students can apply different elementary row operations even though the final answer is the same. However, we can get different forms of REF.

3. Suppose $A = \begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}$. Find the values of a so that $rank(A) = 1$. What will be the answer to make $rank(A) = 2$? How about for $rank(A) = 3$?

Solution:

$$\begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix} \xrightarrow[R_3 \to R_3 - aR_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & a \\ 0 & a-1 & 1-a \\ 0 & 1-a & 1-a^2 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 1 & a \\ 0 & a-1 & 1-a \\ 0 & 0 & 2-a-a^2 \end{pmatrix}$$

i) $rank(A) = 1 \Leftrightarrow a-1 = 1-a = 0,\ 2-a-a^2 = 0 \Leftrightarrow a = 1,\ a = 1, -2 \Leftrightarrow a = 1$

ii) $rank(A) = 2 \Leftrightarrow a-1 \neq 0\ \&\ 2-a-a^2 = 0 \Leftrightarrow a \neq 1 \&\ a = 1, -2 \Leftrightarrow a = -2$

iii) $rank(A) = 3 \Leftrightarrow a-1 \neq 0\ \&\ 2-a-a^2 \neq 0 \Leftrightarrow a \neq 1 \&\ a \neq 1, -2 \Leftrightarrow a \in R/\{1, -2\}$

1.6 Determinants of Matrices and Inverse Computations

1.6.1 Determinant of Square Matrices

Why we study the determinant of a matrix? The determinant of a square matrix A is a scalar that is associated with A and denoted by $\det A$ or $|A|$. The determinant of a matrix can be negative, zero or positive.

a) Determinants of 1×1 Matrices

The determinant of a 1×1 matrix $A=(a_{11})$ is defined as $\det A = a_{11}$.

b) Determinant of 2×2 Matrices: If $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is any 2×2 matrix, then the determinant of A is defined as $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$.

For example, if $A=\begin{pmatrix} 3 & 5 \\ 2 & 7 \end{pmatrix}$, then $\det A = 21-10=11$.

c) Determinant of 3×3 Matrices: Let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Then,

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: Let $A=\begin{pmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 9 \end{pmatrix}$.

Then, $\det A = 2\begin{vmatrix} 4 & 5 \\ 5 & 9 \end{vmatrix} - 3\begin{vmatrix} 3 & 5 \\ 4 & 9 \end{vmatrix} + 4\begin{vmatrix} 3 & 4 \\ 4 & 5 \end{vmatrix} = 22-21-4=-3$

1.6.2 Inverse and Condition for Existence of Inverse

Singular and Non-singular Matrices:

Let A be any square matrix . Then, A is said to be

i) **Non – Singular** matrix if $\det A \neq 0$

ii) **Singular** matrix if $\det A = 0$

Examples:

1. Let $A=\begin{pmatrix} 3 & 4 \\ 2 & 5 \end{pmatrix}$, $B=\begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$. Here, $\det A = 7 \neq 0, \det B = 0$.

So, A is **non-singular** but B is **singular**.

2. Find the value of x for which the matrix $A=\begin{pmatrix} 1 & x+1 & 0 \\ 1 & 0 & -1 \\ 1 & 4 & x \end{pmatrix}$ is *singular*.

Solution: By definition A is singular if and only if $\det A = 0$.

So, $\det A = 4-(x+1)^2 = 0 \Rightarrow x = 1, -3$

Inverse of a matrix: We say that anon-singular matrix A is has inverse if there exists a square matrix B such that $AB = BA = I$ where I is identity matrix. With this condition, A and B are said to be inverse of each other. The inverse of A is denoted by A^{-1} and thus $A^{-1}A = A^{-1}A = I$.

Condition for existence of inverse: Matrices that have inverses are said to be invertible matrices. As we see from the definition of inverse, only non-singular matrices are i have inverse or only non-singular matrices are invertible.

Properties of inverse: Suppose A and B are non-singular matrices of size $n \times n$ and $k \neq 0$ is any scalar. Then,

i) The inverse of A is unique.

ii) $(A^{-1})^{-1} = A$ iii) $(kA)^{-1} = \frac{1}{k}A^{-1}$ iv) $(A^t)^{-1} = (A^{-1})^t$

v) The product AB is non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

1.6.3 Ad-joint Method of Inverse Computations

Sub-matrices of a matrix: A matrix formed by covering (deleting) a row and column of a square matrix A is called **sub matrix** of A. A sub-matrix obtained by covering (deleting) the i^{th} row and the j^{th} column of A is denoted by A_{ij}.

Example: Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 9 & 6 \\ 5 & 8 & 3 \end{pmatrix}$. Then, $A_{12} = \begin{pmatrix} 2 & 6 \\ 5 & 3 \end{pmatrix}$ is a sub matrix of A obtained by deleting the first row and the second column of A. In this way you can form different sub-matrices for a given matrix.

Minors, Cofactors and Ad-joint of a matrix:

Let $A = (a_{ij})_{n\times n}$ and let A_{ij} denote the sub-matrix of A obtained by deleting i^{th} row and j^{th} column of A. Then,

1st -Minor: $M_{ij} = \det(A_{ij}) = |A_{ij}|$ is called the **minor** of the entry a_{ij}.

2nd -Cofactor: $c_{ij} = (-1)^{i+j} M_{ij}$ is called **cofactor** of a_{ij}.

In general, a square matrix of size $n \times n$ has n^2 cofactors.

3rd -Cofactor Matrix of A: The matrix $C = (c_{ij})_{n\times n}$ formed by using the cofactors of A is called cofactor matrix of A.

4th -Ad-joint Matrix of A: The transpose of the cofactor matrix C of A is called **ad-joint** of A and denoted by $adjA = C^t$.

5th –Inverse using ad joint Method: $A^{-1} = \dfrac{1}{\det A} adj\ A$

The method of finding the inverse of a given matrix using the above procedures is known as ad-joint method.

General Formula of Inverse: If we apply the adjoint method for 2 by 2 matrices, we can get the following general formula. The inverse of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is by

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$ where $\det A = ad - bc \neq 0$.

Examples:

1. Given $a)\ A=\begin{pmatrix}1 & -2 & 0\\ -2 & 3 & 1\\ 0 & 1 & 0\end{pmatrix}$ $\quad b)\ A=\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$ $\quad c)\ A=\begin{pmatrix}1 & 2 & 3\\ -1 & 0 & 0\\ 0 & 3 & 4\end{pmatrix}$

Determine i) Whether A is invertible (has inverse) or not

ii) All minors and all cofactors of A

iii) The cofactor matrix and ad joint of A

vi) The inverse of A

Solution:

i) Determinant of A : $\det A = \begin{vmatrix}3 & 1\\ 1 & 0\end{vmatrix} - 2.\begin{vmatrix}-2 & 1\\ 0 & 0\end{vmatrix} + 0.\begin{vmatrix}-2 & 3\\ 0 & 1\end{vmatrix} = -1 \neq 0$

ii) Minors of A : $M_{ij} = \det(A_{ij}) = |A_{ij}|$

$M_{11} = |A_{11}| = \begin{vmatrix}3 & 1\\ 1 & 0\end{vmatrix} = -1,\ M_{12} = |A_{12}| = \begin{vmatrix}-2 & 1\\ 0 & 0\end{vmatrix} = 0,\ M_{13} = |A_{13}| = \begin{vmatrix}-2 & 3\\ 0 & 1\end{vmatrix} = -2$

$M_{21} = |A_{21}| = \begin{vmatrix}-2 & 0\\ 1 & 0\end{vmatrix} = 0,\ M_{22} = |A_{22}| = \begin{vmatrix}1 & 0\\ 0 & 0\end{vmatrix} = 0,\ M_{23} = \begin{vmatrix}1 & -2\\ 0 & 1\end{vmatrix} = 1$

$M_{31} = |A_{31}| = \begin{vmatrix}-2 & 0\\ 3 & 1\end{vmatrix} = -2,\ M_{32} = |A_{32}| = \begin{vmatrix}1 & 0\\ -2 & 1\end{vmatrix} = 1,\ M_{33} = \begin{vmatrix}1 & -2\\ -2 & 3\end{vmatrix} = -1$

Cofactors of A : $C_{ij} = (-1)^{i+j}|A_{ij}| = (-1)^{i+j} M_{ij}$

$C_{11} = (-1)^2 M_{11} = -1,\quad C_{12} = (-1)^3 M_{12} = 0,\quad C_{13} = (-1)^4 M_{13} = -2$

$C_{21} = (-1)^3 M_{21} = 0,\quad C_{22} = (-1)^4 M_{22} = 0,\quad C_{23} = (-1)^5 M_{23} = -1$

$C_{31} = (-1)^4 M_{31} = -2,\quad C_{32} = (-1)^5 M_{32} = -1,\quad C_{33} = (-1)^6 M_{33} = -1$

iii) Cofactor matrix of A : $C = \begin{pmatrix}c_{11} & c_{12} & c_{13}\\ c_{21} & c_{22} & c_{23}\\ c_{31} & c_{32} & c_{33}\end{pmatrix} = \begin{pmatrix}-1 & 0 & -2\\ 0 & 0 & -1\\ -2 & -1 & -1\end{pmatrix}$

iv) Ad joint matrix of A : $adjA = C' = \begin{pmatrix}-1 & 0 & -2\\ 0 & 0 & -1\\ -2 & -1 & -1\end{pmatrix}$

v) The inverse of A : $A^{-1} = \frac{1}{\det A} adj\ A = -1.\begin{pmatrix} -1 & 0 & -2 \\ 0 & 0 & -1 \\ -2 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 1 \end{pmatrix}$

b) Using similar procedures, we get $A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (Here, $A^{-1} = A$)

c) Here also we have $A^{-1} = \begin{pmatrix} 0 & -1 & 0 \\ -4 & -4 & 3 \\ 3 & 3 & -2 \end{pmatrix}$

2. If $A = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ is the inverse of $B = \begin{pmatrix} x & 3 \\ 3 & y \end{pmatrix}$, find x and y

Solution: By the above general inverse formula, $A^{-1} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$.

Then, $B = A^{-1} \Rightarrow \begin{pmatrix} x & 3 \\ 3 & y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \Rightarrow x = 4, y = -2$.

3. For what values of a and b does the matrix $A = \begin{pmatrix} 0 & a & a \\ 3 & -2 & 1 \\ b & 0 & 2 \end{pmatrix}$

a) Will have inverse (it will be non-singular)?

b) Will have no inverse (it will be singular)?

Solution:

Here, $\det A = \begin{vmatrix} 0 & a & a \\ 3 & -2 & 1 \\ b & 0 & 2 \end{vmatrix} = -a\begin{vmatrix} 3 & 1 \\ b & 2 \end{vmatrix} + a\begin{vmatrix} 3 & -2 \\ b & 0 \end{vmatrix} = -6a + 3ab = 3a(b-2)$

a) We need matrix A to have inverse (that means to be non-singular).
So, $\det A \neq 0 \Rightarrow 3a(b-2) \neq 0 \Rightarrow 3a \neq 0$ and $b - 2 \neq 0 \Rightarrow a \neq 0$ and $b \neq 2$

b) We need matrix A to have no inverse (to be singular).
Hence, $\det A = 0 \Rightarrow 3a(b-2) = 0 \Rightarrow 3a = 0$ or $b - 2 = 0 \Rightarrow a = 0$ or $b = 2$

4. Find c such that the matrix $A-cI_2$ is not invertible where $A=\begin{pmatrix}4 & 6\\ -1 & -3\end{pmatrix}$

Solution: Here, $A-cI_2=\begin{pmatrix}4-c & 6\\ -1 & -3-c\end{pmatrix}$ is not invertible if $\det(A-cI_2)=0$

So, $\det(A-cI)=0\Rightarrow \det\begin{pmatrix}4-c & 6\\ -1 & -3-c\end{pmatrix}=0\Rightarrow (c-3)(c+2)=0\Rightarrow c=3,-2$

5. If the ad joint of a matrix is $\begin{pmatrix}2 & -3\\ 4 & 5\end{pmatrix}$, find the matrix.

Solution: Suppose $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$.Then, calculate the ad joint of A.

a) Minors of A:

$M_{ij}=\det(A_{ij})=|A_{ij}|; M_{11}=|A_{11}|=d,\ M_{12}=|A_{12}|=c,\ M_{21}=b, M_{22}=a$

b) Cofactors of A: $C_{ij}=(-1)^{i+j}|M_{ij}|;\ c_{11}=d, c_{12}=-c, c_{21}=-b, c_{22}=a$

c) Cofactor matrix of A: $C=\begin{pmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{pmatrix}=\begin{pmatrix}d & -c\\ -b & a\end{pmatrix}$

d) Ad joint matrix of A: $adjA=C'=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$. Since the adjoint of a matrix is unique, we have $adjA=\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}=\begin{pmatrix}2 & -3\\ 4 & 5\end{pmatrix}\Rightarrow d=2, c=-4, b=3, a=5$

6. Let $A=\begin{pmatrix}3 & -5\\ -2 & 3\end{pmatrix}$ and $B=\begin{pmatrix}0 & -5\\ 2 & -4\end{pmatrix}$. Find Y such that $Y+A^{-1}-B=I_2$.

Solution: Note that I_2 means identity matrix of order 2.Besides, using the inverse formula of 2×2 matrices, we have $A^{-1}=\begin{pmatrix}-3 & -5\\ -2 & -3\end{pmatrix}$.

So, $Y+A^{-1}-B=I_2\Rightarrow Y=I_2-A^{-1}+B$

$$=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}-3 & -5\\ -2 & -3\end{pmatrix}+\begin{pmatrix}0 & -5\\ 2 & -4\end{pmatrix}=\begin{pmatrix}4 & 0\\ 4 & 0\end{pmatrix}$$

1.6.4 Basic Properties of Determinants

Let **A** and **B** be $n \times n$ matrices. Then, we have the following impo... properties about determinant.

Property-1: Transpose and Inverse Properties of determinants

a) **Transpose Property:** $\det \mathbf{A}' = \det \mathbf{A}$.

b) **Inverse Property:** $\det A^{-1} = \dfrac{1}{\det A}$ (provided A is invertible)

Property-2: Product and Power Properties:

a) **Product Property:** $\det(\mathbf{AB}) = \det \mathbf{A}.\det \mathbf{B}$ provided AB is compatible

b) **Power Property:** $\det(\mathbf{A}^2) = (\det \mathbf{A})^2, \det(\mathbf{A}^m) = (\det \mathbf{A})^m$.

Property-3: Ad-joint Property: $\det(adj\mathbf{A}) = (\det \mathbf{A})^{n-1}$ where n is order of A.

Property-4: Scalar multiple Property

a) $\det(k\mathbf{A}) = k^n \det \mathbf{A}$ if **A** is multiplied by the scalar k.

b) $\det(\mathbf{B}) = t.\det \mathbf{A}$ if matrix **B** is obtained from **A** by multiplying only one row (column) of **A** with a scalar t.

Property-5: Properties leading to zero determinant

a) If at least one row or column of **A** is zero, then $\det \mathbf{A} = 0$.

b) If one row (column) of **A** is a multiple of another row (column),or if there are two equal row or columns, then $\det \mathbf{A} = 0$.

Property-6: Special Matrix Property:

The determinant of any triangular, diagonal or scalar matrix is the product of its diagonal entries. Particularly, the determinant of any identity matrix is 1.

Property-7: Effects of elementary operations

a) **Swamping or interchanging:** If matrix **B** is obtained by interchanging two adjacent rows or columns of **A**, then $\det(\mathbf{B}) = -\det \mathbf{A}$.

b) **Adding (subtracting) rows or columns:** If **B** is obtained from **A** by adding (subtracting) a scalar multiple of a row (column) into another row (column), $\det(\mathbf{B}) = \det \mathbf{A}$.

Property-8: Equality Property

If A and B are equal, then they have equal determinants

Examples:

1. If A and B are 3×3 matrices such that $\det(A)=3, \det B=2$, then using properties of determinant, find $\det(AB), \det(A^2), \det(3B')$ and $\det(2A'B)$.

Solution: Apply the above properties of determinants one by one.

i) By product property: $\det(AB)=\det(A)\det(B)=3(2)=6$

ii) By power property: $\det(A^2)=(\det A)^2=3^2=9$

iii) By transpose property: $\det(3B')=3^3\det(B')=27\det B=27(2)=54$

iv) Property 1 and 2: $\det(A^2B')=\det(A^2).\det(B')=(\det A)^2\det B=9(2)=18$

2. If $A=\begin{pmatrix}2x & 3x\\ 3x & 5x\end{pmatrix}$, then for what values of x is $\det(2A')=x^4$?

Solution: Here, $\det(A)=10x^2-9x^2=x^2$. Besides, using $\det A'=\det A$,

$\det(2A')=x^4 \Rightarrow 4\det(A')=4\det A=x^4 \Rightarrow 4x^2=x^4 \Rightarrow x=0, x=\pm2$

3. If A and B are 3×3 matrices with $AB=2I$ and $\det(B)=6$, find $\det(3A')$.

Solution: Recall: $\det(A')=\det(A), \det(AB)=\det(A)\det(B), \det(I)=1$.

Hence, $AB=2I \Rightarrow \det(AB)=\det(2I) \Rightarrow \det(A)\det(B)=2^3\det(I)=8$

$$\Rightarrow \det(A)=\tfrac{8}{\det B}=\tfrac{8}{6}=\tfrac{4}{3} \Rightarrow \det(A')=\det(A)=\tfrac{4}{3}$$

Therefore, $\det(3A')=3^3\det(A')=27\det(A)=27(\frac{4}{3})=36$.

4. Let A be 3×3 scalar matrix with $\det A=64$. Then, find matrix A.

Solution: A 3×3 scalar matrix is of the form $A=\begin{pmatrix}a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & a\end{pmatrix}$.

Given that $\det A=64$. Then, $\det(A)=a^3=64 \Rightarrow a=4$.

5. If A is 3×3 matrix with $\det(-2A)'=48$, then find $\det(\frac{1}{3}A)$.

Solution: By transpose and scalar multiple property, $\det(-2A)=(-2)^3\det A$.

Thus, $\det(-2A)=(-2)^3\det A=48 \Rightarrow \det A=-6$.

Finally, we have $\det(\frac{1}{3}A)=\left(\frac{1}{3}\right)^3\det A=\frac{-6}{27}=-\frac{2}{9}$.

6. Let $A=\begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$ and $\det A=-10$. Then, find

a) $\begin{vmatrix} a-3d & d & g \\ c-3f & f & i \\ b-3e & e & h \end{vmatrix}$ b) $\begin{vmatrix} 3a & 12d & 3g \\ c & 4f & i \\ 2b & 8e & 2h \end{vmatrix}$ c) $\begin{vmatrix} 6a & 3b & 12c \\ 2d & e & 4f \\ 10g & 5h & 20i \end{vmatrix}$

d) $\begin{vmatrix} 6a & 3b & 12c \\ 2d & e & 4f \\ 10g & 5h & 20i \end{vmatrix}$ e) $\begin{vmatrix} 6a+g & 30b+5h & 6c+i \\ g & 5h & i \\ d & 5e & f \end{vmatrix}$

Solution:

a) By property 10 and 11, $\begin{vmatrix} a-3d & d & g \\ c-3f & f & i \\ b-3e & e & h \end{vmatrix}=\begin{vmatrix} a & d & g \\ c & f & i \\ b & e & h \end{vmatrix}=-\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}=10$

b) $\begin{vmatrix} 3a & 12d & 3g \\ c & 4f & i \\ 2b & 8e & 2h \end{vmatrix}=3\begin{vmatrix} a & 4d & g \\ c & 4f & i \\ 2b & 8e & 2h \end{vmatrix}=-3(4)(2)\begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}=-24(-10)=240$

c) $\begin{vmatrix} 6a & 3b & 12c \\ 2d & e & 4f \\ 10g & 5h & 20i \end{vmatrix}=3(2)\begin{vmatrix} a & b & 4c \\ d & e & 4f \\ 5g & 5h & 20i \end{vmatrix}=3(2)(4)(5)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}=120(-10)=-1200$

d) $\begin{vmatrix} 4a & 6b & 10c \\ 2d & 3e & 5f \\ 8g & 12h & 20i \end{vmatrix}=2(2)(3)\begin{vmatrix} a & b & 5c \\ d & e & 5f \\ 4g & 4h & 20i \end{vmatrix}=2(2)(3)(4)(5)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}=-2400$

e) $\begin{vmatrix} 6a+g & 30b+5h & 6c+i \\ g & 5h & i \\ d & 5e & f \end{vmatrix}=5.\begin{vmatrix} 6a+g & 6b+h & 6c+i \\ g & h & i \\ d & e & f \end{vmatrix}=5.\begin{vmatrix} 6a & 6b & 6c \\ g & h & i \\ d & e & f \end{vmatrix}$

$$=5(6)\begin{vmatrix} a & b & c \\ g & h & i \\ d & e & f \end{vmatrix}=30(-1)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}=300$$

7. If $\begin{bmatrix}1 & -2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} B \begin{bmatrix}5 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = 5I_3$, what is $\det B$?

Solution: By equality property, we can take determinant both sides.

That is $\begin{vmatrix}1 & -2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} |B| \begin{vmatrix}5 & 0 & 0\\ -2 & 1 & 0\\ 0 & 0 & 1\end{vmatrix} = \det(5I_3) \Rightarrow 5\det B = 5^3 \det I_3 \Rightarrow \det B = 25$

8. In each of the following, find the required determinants:

a) If A and B are 4×4 non-singular with $\det B = -3$,find $\det(2AB'A^{-1})$

b) If A is an invertible matrix of order 3, then find $\det(AA^{-1} + 2(I_3)^5)$.

c) Let $B = 2A$ and $C = 4B^{-1}$ where A is 2×2 with $\det A = 3$. Find $\det C$.

d) If A is a 2×2 matrix with $B = 2A$ and $C = 3B^{-1}$. Find $\det(2ACB'A^{-1})$.

Solution:

a) Here, $\det(2AB'A^{-1}) = 2^4 \det A \det B \frac{1}{\det A} = -48$.

b) Since $AA^{-1} = I_3$, $2(I_3)^5 = 2I_3$, we have

$\det(AA^{-1} + 2(I_3)^5) = \det(I_3 + 2I_3) = \det(3I_3) = 3^3\det(I_3) = 27$

c) Here, $B = 2A \Rightarrow \det B = \det(2A) = 4\det A = 12$. So,

$C = 4B^{-1} \Rightarrow \det C = \det(4B^{-1}) = 4^2 \det B^{-1} = \frac{16}{\det B} = \frac{4}{3}$

d) $\det(2ACB'A^{-1}) = 2^2 \det A.\det C.\det B'.\det A^{-1} = 36$

9. Let $B = \begin{pmatrix}1 & 0 & 0\\ 3 & -3 & 0\\ 4 & 6 & 5\end{pmatrix}$. If $\det A = \frac{2}{9}$, then find $\det(\frac{2}{3}A^{-1}B')$.

Solution: Recall: $\det(kA) = k^n\det(A)$, $\det A^{-1} = \frac{1}{\det A}$, $\det(B') = \det B$

Since B is lower triangular, $\det B = (1)(-3)(5) = -15 \Rightarrow \det B' = -15$.

Therefore, $\det(\frac{2}{3}A^{-1}B') = (\frac{2}{3})^3\det(A^{-1})\det(B') = \frac{8}{27}\cdot\frac{9}{2}\cdot(-15) = -20$

10. Find $\det[5B'A^2(3B)^{-1}]$ if A and B are 2 by 2 matrices where $A = 6A^{-1}$.

Solution: First, by equality property, $A = 6A^{-1}$.

So, $\det A = \det(6A^{-1}) \Rightarrow \det A = 6^2 \det(A^{-1}) \Rightarrow \det A = \frac{36}{\det A} \Rightarrow (\det A)^2 = 36$

Therefore, we have

$$\det[5B'A^2(3B)^{-1}] = 5^2 \det B.(\det A)^2(\tfrac{1}{3})^2 \det B^{-1}$$

$$= 25 \det B(36)\tfrac{1}{9}\tfrac{1}{\det B} = 25.(36).\tfrac{1}{9} = 100$$

11*. Let $X = \begin{pmatrix} 1 & 2 & -4 \\ 0 & -3 & 0 \\ -2 & 6 & -5 \end{pmatrix}$. Find a matrix Y so that $3X - 2Y$ is a scalar matrix with $\det(4Y - 6X) = 8$.

Solution: Since $3Y - 2X$ is a scalar matrix, $3X - 2Y = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$.

So, from property of determinant, $\det(3X - 2Y) = a^3$. On the other hand, observe that $4Y - 6X = -2(3X - 2Y)$. Then, taking determinant both sides, we have $\det(4Y - 6X) = \det[-2(3X - 2Y)] = (-2)^3 \det(3X - 2Y) = -8\det(3X - 2Y)$.

But we are given that $\det(4Y - 6X) = 8$.

Hence, $\det(4Y - 6X) = 8 \Rightarrow -8\det(3X - 2Y) = 8 \Rightarrow \det(3X - 2Y) = -1$.

Equating $\det(3X - 2Y) = a^3$ and $\det(3X - 2Y) = -1$, $a^3 = -1 \Rightarrow a = -1$.

Finally, we get the required matrix Y as follow:

$$3X - 2Y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow 2Y = 3\begin{pmatrix} 1 & 2 & -4 \\ 0 & -3 & 0 \\ -2 & 6 & -5 \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Rightarrow Y = \frac{1}{2}\begin{pmatrix} 4 & 6 & -12 \\ 0 & -8 & 0 \\ -6 & 18 & -14 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -6 \\ 0 & -4 & 0 \\ -3 & 9 & -7 \end{pmatrix}$$

12. Suppose A, B and C are 3×3 non-singular matrices where $\det A = 4$ and $\det B = 5$. Then, find $\det[A^{-2}adj(8B)']$ and $\det[3(6AC)^{-1}(4BC)']$.

Solution: Since A and B are 3×3, $adj(3B)'$ is also 3×3.

Recall the property that for any $n\times n$ matrix A, $\det(adjA) = (\det A)^{n-1}$.

i) $\det[A^{-2}adj(8B)'] = \det A^{-2}\det[adj(8B)'] = 8^3\det A^{-2}\det[adjB']$

$$= \frac{8^3\det[adjB']}{\det A^2} = \frac{8^3(\det B)^2}{(\det A)^2} = \frac{8^3(25)}{16} = 800$$

ii) $\det[3(6AC)^{-1}(4BC)'] = 3^3\det(6AC)^{-1}\det(4BC)' = \dfrac{27\det(4BC)}{\det(6AC)}$

$$= \frac{27(4^3)\det B.\det C}{6^3\det A.\det C} = \frac{8\det B}{\det A} = \frac{8(5)}{4} = 10$$

13. If $(\frac{1}{3}A)^{-1} = \begin{pmatrix}1 & 2\\ 3 & 5\end{pmatrix}$, then find matrix A.

Solution: Recall that $(A^{-1})^{-1} = A$ (The 3rd property of inverse given above).

That is $\left((\frac{1}{3}A)^{-1}\right)^{-1} = \begin{pmatrix}1 & 2\\ 3 & 5\end{pmatrix}^{-1} \Rightarrow \frac{1}{3}A = \begin{pmatrix}-5 & 2\\ 3 & -1\end{pmatrix} \Rightarrow A = \begin{pmatrix}-15 & 6\\ 9 & -3\end{pmatrix}$

14. Let $A = \begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}$. Find the matrix X such that $AX - A' = 2A$.

Solution: Here, $AX - A' = 2A \Rightarrow AX = A' + 2A \Rightarrow X = A^{-1}[A' + 2A]$

So, $X = A^{-1}[A' + 2A] = \begin{pmatrix}-3 & 2\\ 2 & -1\end{pmatrix}\left[\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix} + 2\begin{pmatrix}1 & 2\\ 2 & 3\end{pmatrix}\right] = \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$

1.7 System of Linear Equations and Methods of Solutions

1.7.1 Introduction

A finite set of linear equations in the variables $x_1, x_2, x_3, \cdots, x_n$ is called *a system of linear equation*. i.e. A system of the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{1n}x_n = b_2 \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ \quad\vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m \end{cases}$$

is called *system of m linear equations in n variables.*

1.7.2 Matrix Equation of a Linear System

The above system can be written equivalently in matrix form as

$$\begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \quad \vdots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

In short, it is written in matrix form $AX = B$ where

$$A = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \ldots a_{2n} \\ \vdots & \vdots \quad \vdots \\ a_{m1} & a_{m2} \ldots a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Here, A is called *coefficient matrix* of the system, X is called variable matrix, B is called matrix of constants and (A/B) is called *augmented matrix.*

1.7.3 Solutions of a system and Number of Solutions

Definition: The values of the variables $x_1, x_2, x_3, \cdots, x_n$ that satisfy all the equations of the system are called solutions. A given linear system of equations may have a *unique solution (*exactly one solution*), many solutions* (infinite solutions) or *no solutions* (if there are no values for the variables satisfying all the linear equations of the system). We say that the system is *consistent* if it has at least one solution. Otherwise, it is called *inconsistent*.

A system that has a unique solution is called *independent system* and a system that has more than one solution is called *dependent system*.

Theorem (The Rank Test for number of solutions):

Suppose $AX = B$ is a linear equation where A is an $m \times n$ matrix. Then,

a) If $rank(A/B) = rank(A) = n$, then the system has a unique solution and the system is consistent and independent.

b) If $rank(A/B) = rank(A) = r < n$, then the system has more than one solution and the system is consistent and dependent.

c) If $rank(A/B) \neq rank(A)$, then the system has no solution (it is inconsistent)

Examples:

1. Determine the values of the constants k and t so that the following systems will have Unique, Many or No solution.

$$a)\ \begin{cases} x + ky = 7 \\ x + 3y = t \end{cases} \qquad b)\ \begin{cases} x + 5y + 2z = 2 \\ x + ky + z = 3 \\ x + 5y + kz = 2t \end{cases}$$

Solution: To determine the constants based on the number of solutions, first change the augmented matrix into row-echelon form and apply the rank test.

$$a)\ \left(\begin{array}{cc|c} 1 & k & 7 \\ 1 & 3 & b \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{cc|c} 1 & k & 7 \\ 0 & 3-k & b-7 \end{array}\right)$$

Unique solution if and only if $rank(A/B) = rank(A) = n$ where $n = 3$ (number of columns of A). But, $rank(A/B) = rank(A) = 2 \Leftrightarrow 3 - a \neq 0 \Leftrightarrow a \neq 3$

Hence, the system will have a unique solution if and only if $a \neq 3, b \in R$.

More than one solution if $rank(A/B) = rank(A) < n$. But,
$rank(A/B) = rank(A) < 2 \Leftrightarrow 3-a=0,\ b-7=0 \Leftrightarrow a=3,\ b=7$.
Hence, the system will have more than one solution if and only if $a=3,\ b=7$.
No solution if and only if $rank(A/B) \neq rank(A)$. But,
$rank(A/B) \neq rank(A) \Leftrightarrow 3-a=0,\ b-7\neq 0 \Leftrightarrow a=3,\ b\neq 7$.

$$b)\left(\begin{array}{ccc|c} 1 & 5 & 2 & 2 \\ 1 & k & 1 & 3 \\ 1 & 5 & k & 2t \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 5 & 2 & 2 \\ 0 & k-5 & -1 & 1 \\ 0 & 0 & k-2 & 2t-2 \end{array}\right)$$

Unique solution if and only if $rank(A/B) = rank(A) = n$ where $n=2$ (number of columns of A). But, $rank(A/B) = rank(A) = 3 \Leftrightarrow k-2\neq 0 \Leftrightarrow k\neq 2$
Hence, the system will have a unique solution if and only if $k\neq 2, t\in R$.
More than one solution if $rank(A/B) = rank(A) < n$. But,
$rank(A/B) = rank(A) < 3 \Leftrightarrow k-2=0,\ 2t-2=0 \Leftrightarrow k=2,\ t=1$.
Hence, the system will have more than one solution if and only if $k=2,\ t=1$.
No solution if $rank(A/B) \neq rank(A) \Leftrightarrow k-2=0, 2t-2\neq 0 \Rightarrow k=2, t\neq 1$.

2. If the augmented matrix of a certain system is known to be $\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right)$

Determine the values of a and b so that the system will have unique solution, one parametric solutions, two parametric solutions or No solution.
Solution:

$$\left(\begin{array}{ccc|c} a & 0 & b & 2 \\ a & a & 4 & 4 \\ 0 & a & 2 & b \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & a & 2 & b \end{array}\right) \xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccc|c} a & 0 & b & 2 \\ 0 & a & 4-b & 2 \\ 0 & 0 & b-2 & b-2 \end{array}\right)$$

Unique solution if and only $rank(A/B) = rank(A) = 3$.
But $rank(A/B) = rank(A) = 3 \Leftrightarrow a\neq 0, b-2\neq 0 \Leftrightarrow a\neq 0, b\neq 2$
One parametric solution (two of the variables are leading variables or only one of the variables is free) if $rank(A/B) = rank(A) = 2 < 3 \Rightarrow a\neq 0, b=2$.
Two parametric solution (only one of the variables is leading variable or there are two free variables).In this case, $rank(A/B) = rank(A) = 1 < 3 \Rightarrow a=0, b=2$.
(Please read about Free and Leading variables)

1.7.4 Different Methods of Solving Linear Systems

A) Matrix Inverse Method

A given system of simultaneous equation can be written in matrix form as $AX = B$. Suppose the coefficient matrix A is invertible. Then, if we multiply both sides by A^{-1}, $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow X = A^{-1}B$. Therefore, the matrix equation of the form $AX = B$ has a unique solution given by $X = A^{-1}B$ whenever A^{-1} exists. This method is known as *matrix inverse method.*

Examples: Solve the following systems using matrix inverse method.

a) $AX = B$ where $A^{-1} = \begin{pmatrix} 2 & -3 & 4 \\ 3 & 0 & 2 \\ -1 & 1 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ b) $\begin{cases} 2x_1 + x_2 = 13 \\ x_1 - 3x_2 = -11 \end{cases}$

Solution:

a) $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}B = \begin{pmatrix} 2 & -3 & 4 \\ 3 & 0 & 2 \\ -1 & 1 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 12 \end{pmatrix}$. Hence, the solutions are $x_1 = 1,\ x_2 = 7,\ x_3 = 12$.

b) Here, $A = \begin{pmatrix} 2 & 1 \\ 1 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 13 \\ -11 \end{pmatrix}$. Then, using the formula for the inverse of 2×2 matrices, we have $A^{-1} = \begin{pmatrix} \frac{3}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{2}{7} \end{pmatrix}$.

So, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A^{-1}B = \begin{pmatrix} \frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & -\frac{2}{7} \end{pmatrix}\begin{pmatrix} 13 \\ -11 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

B) Gaussian Elimination Method

The process of solving system of linear equations by changing the augmented matrix of the system in to its row-echelon form and then solving for the variables by back substitution is known as Gaussian-elimination method. The main advantage of this method is that it is applicable to any system of linear equations.

Procedures: To solve the system $AX = B$, using Gaussian elimination method:

Step-1: Form the augmented matrix (A/B) of the given system.

Step-2: Transform the augmented matrix (A/B) into REF say $(A/B) \to (U/C)$.

Step-3: Form a new system of linear equations using $UX = C$ and solve for the variables using back substitution.

Note that while using this method if you get an equation of the form $0 = 0$, it means the system has many solutions and if you get an equation of the form $0 = r \neq 0$, it means the system has no solution.

Examples: Solve the following systems using Gaussian-Elimination method

a) $\begin{cases} x_1 + x_2 - x_3 + 2x_4 = 5 \\ x_1 + 2x_2 - x_3 + 2x_4 = 3 \\ 2x_1 + 2x_2 - 2x_3 + 5x_4 = 11 \end{cases}$ b) $\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ 2x_1 + 2x_2 + 2x_4 = 4 \\ -x_1 - x_3 + x_4 = 3 \end{cases}$ c) $\begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_1 + 2x_2 + 2x_3 + 2x_4 = 1 \\ x_1 + 2x_2 + 3x_3 + 3x_4 = 1 \end{cases}$

d) $\begin{cases} x_1 - 3x_2 - x_3 + 2x_4 = 5 \\ x_1 - x_2 - 3x_3 + 2x_4 = 9 \\ 3x_1 - 5x_2 - 7x_3 + 3x_4 = 11 \end{cases}$ e) $\begin{cases} x_1 + 2x_2 + 5x_3 = 1 \\ 2x_1 + 3x_2 + 8x_3 = 1 \\ -x_1 + x_2 + 2x_3 = 2 \end{cases}$ f) $\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 4 \\ -2x_1 + 4x_2 - x_3 - 3x_4 = -2 \\ 6x_1 - 12x_2 + x_3 + 19x_4 = 5 \end{cases}$

Solution:

a) $\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 5 \\ 1 & 2 & -1 & 2 & 3 \\ 2 & 2 & -2 & 5 & 11 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 5 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$

Now form a new system using $UX = C$ and apply back-substitution.

That is $UX = C \Rightarrow \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_2 - x_3 + 2x_4 = 5 \\ x_2 = -2 \\ x_4 = 1 \end{cases}$

Then, $x_1 + x_2 - x_3 + 2x_4 = 5 \Rightarrow x_1 - x_3 = 5 \Rightarrow x_1 = x_3 + 5$. Hence, letting $x_3 = t$, the general solution becomes $x_1 = 5 + t, x_2 = -2, x_3 = t, x_4 = 1, t \in R$.

b) $\begin{pmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 2 & 2 & 0 & 2 & | & 4 \\ -1 & 0 & -1 & 1 & | & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 0 & -2 & 2 & 2 & | & 2 \\ 0 & 2 & -2 & 1 & | & 4 \end{pmatrix} \xrightarrow{R_3 \to R_3 + R_2} \begin{pmatrix} 1 & 2 & -1 & 0 & | & 1 \\ 0 & -2 & 2 & 2 & | & 2 \\ 0 & 0 & 0 & 3 & | & 6 \end{pmatrix}$

Now form a new system using $UX = C$ and apply back-substitution.

That is $UX = C \Rightarrow \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 - x_3 = 1 \\ -2x_2 + 2x_3 + 2x_4 = 2 \\ 3x_4 = 6 \end{cases}$

Hence, by letting, $x_3 = t$, we get $x_4 = 2$, $x_2 = t+1, x_1 = -t-1, t \in R$

c) $\begin{pmatrix} 1 & 1 & 1 & 1 & | & 1 \\ 1 & 2 & 2 & 2 & | & 1 \\ 1 & 2 & 3 & 3 & | & 1 \end{pmatrix} \xrightarrow{R_3 \to R_3 - R_2} \begin{pmatrix} 1 & 1 & 1 & 1 & | & 1 \\ 1 & 2 & 2 & 2 & | & 1 \\ 0 & 0 & 1 & 1 & | & 0 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 & 1 & | & 1 \\ 0 & 1 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & 1 & | & 0 \end{pmatrix}$

Now form a new system using $UX = C$ and apply back-substitution.

That is $UX = C \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_2 + x_3 + x_4 = 1 \\ x_2 + x_3 + x_4 = 0 \\ x_3 + x_4 = 0 \end{cases}$

Hence, letting $x_4 = t$, the solution is $\{(x_1, x_2, x_3, x_4)\} = \{(1, 0, -t, t) : t \in R\}$.

d) $\begin{pmatrix} 1 & -3 & -1 & 2 & | & 5 \\ 1 & -1 & -3 & 2 & | & 9 \\ 3 & -5 & -7 & 3 & | & 11 \end{pmatrix} \xrightarrow[\substack{R_3 \to R_3 - 3R_1 \\ R_3 \to R_3 - 2R_2}]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & -3 & -1 & 2 & | & 5 \\ 0 & 2 & -2 & 0 & | & 4 \\ 0 & 0 & 0 & -3 & | & -12 \end{pmatrix}$

Now set up a new systemand applyback substitution.

That is $UX = C \Rightarrow \begin{pmatrix} 1 & -3 & -1 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ -12 \end{pmatrix} \Rightarrow \begin{cases} x_1 - 3x_2 - x_3 + 2x_4 = 5 \\ 2x_2 - 2x_3 = 4 \\ -3x_4 = -12 \end{cases}$

Then, by letting $x_3 = t$, we get $x_2 = t+2, x_1 = 4t+3$

Therefore, the solution set is $\{(x_1, x_2, x_3, x_4)\} = \{(4t+3, t+2, t, 4) : t \in R\}$

C) Gauss-Jordan Elimination Method

The process of solving system of linear equations first by changing the augmented matrix of the system in to its ***reduced-row-echelon*** form and then solving for the variables is known as *Gauss-Jordan elimination method*. The main difference between this method and Gaussian Elimination is that in Gaussian Elimination, we change the augmented matrix into *row-echelon form* whereas in Gauss-Jordan method it is changed into *reduced-row-echelon* form.

Examples: Solve the following systems using Gauss-Jordan elimination Method

$a)\begin{cases} x_1 - x_2 + 2x_3 + x_4 = 2 \\ 2x_1 - 3x_2 + 3x_3 + 2x_4 = 5 \\ 3x_1 - 3x_2 + 6x_3 + 4x_4 = -1 \end{cases}$ $b)\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 3 \\ 2x_1 - 4x_2 + x_3 + x_4 = 2 \\ x_1 - 2x_2 - 2x_3 + 3x_4 = 1 \end{cases}$ $c)\begin{cases} x_1 + 2x_2 + 5x_3 = 4 \\ 2x_1 + 3x_2 + 8x_3 = 7 \\ -x_1 + x_2 + 2x_3 = -1 \end{cases}$

$d)\begin{cases} x_1 - x_2 + x_3 = 0 \\ 3x_1 - 3x_2 = 0 \\ 2x_1 - 2x_2 - 3x_3 = 0 \end{cases}$ $e)\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 2 \\ x_1 + x_2 + x_3 - x_4 = 2 \\ x_1 + x_3 - x_4 = 2 \end{cases}$ $f)\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 3 \\ x_1 + x_2 + x_3 - x_4 = 1 \\ x_1 + x_3 - x_4 = 2 \end{cases}$

Solution:

$$a)\left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 2 & -3 & 3 & 2 & 5 \\ 3 & -3 & 6 & 4 & -1 \end{array}\right) \xrightarrow[R_3\to R_3-3R_1]{R_2\to R_2-2R_1} \left(\begin{array}{cccc|c} 1 & -1 & 2 & 1 & 2 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -7 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1-R_2} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 1 & 1 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -7 \end{array}\right) \xrightarrow[R_2\to -R_2]{R_1\to R_1-R_3} \left(\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 8 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -7 \end{array}\right)$$

Now form a new system using $UX = C$ and apply back-substitution.

That is $UX = C \Rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 8 \\ -1 \\ -7 \end{pmatrix} \Rightarrow \begin{cases} x_1 + 3x_3 = 8 \\ x_2 + x_3 = -1 \\ x_4 = -7 \end{cases}$

Letting $x_3 = t$, we get $x_2 = -1 - t$, $x_1 = 8 - 3t$.

Therefore, the solution is $\{(x_1, x_2, x_3, x_4)\} = \{(8-3t, -1-t, t, -7) : t \in R\}$

$$b)\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 3 \\ 2 & -4 & 1 & 1 & 2 \\ 1 & -2 & -2 & 3 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 & -4 \\ 0 & 0 & -3 & 4 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 3R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 3 \\ 0 & 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & -5 & 10 \end{array}\right) \xrightarrow[R_3 \to \frac{-1}{5}R_3]{R_1 \to R_1 + R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 2 & -1 \\ 0 & 0 & -1 & 3 & -4 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right).$$

$$\xrightarrow[R_2 \to R_2 - 3R_3]{R_1 \to R_1 - 2R_3} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{R_2 \to -R_2} \left(\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

Now form a new system using $UX = C$ and apply back-substitution.

$$\text{That is } UX = C \Rightarrow \begin{pmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} \Rightarrow \begin{cases} x_1 - 2x_2 = 3 \\ x_3 = -2 \\ x_4 = -2 \end{cases}$$

Letting $x_2 = t$, we get $x_1 = 2t + 3$.

Therefore, the solution is $\{(x_1, x_2, x_3, x_4)\} = \{(2t+3, t, -2, -2) : t \in R\}$

$$c)\left(\begin{array}{ccc|c} 1 & 2 & 5 & 4 \\ 2 & 3 & 8 & 7 \\ -1 & 1 & 2 & -1 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 5 & 4 \\ 0 & -1 & -2 & -1 \\ 0 & 3 & 7 & 3 \end{array}\right) \xrightarrow[R_3 \to R_3 + 3R_2]{R_1 \to R_1 + 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 \to -R_2 - 2R_3]{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

Then, from $UX = C$, by back-substitution, we have $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

Hence, the solutions are $x_3 = 0, x_2 = 1, x_1 = 2$.

$$d)\begin{pmatrix}1&-1&1&|&0\\3&-3&0&|&0\\2&-2&-3&|&0\end{pmatrix}\xrightarrow[R_3\to R_3-2R_1]{R_2\to R_2-3R_1}\begin{pmatrix}1&-1&1&|&0\\0&0&-3&|&0\\0&0&-5&|&0\end{pmatrix}$$

$$\xrightarrow[R_1\to R_1+\frac{1}{3}R_2]{R_3\to R_3-\frac{5}{3}R_2}\begin{pmatrix}1&-1&0&|&0\\0&0&-3&|&0\\0&0&0&|&0\end{pmatrix}\xrightarrow{R_2\to-\frac{1}{3}R_2}\begin{pmatrix}1&-1&0&|&0\\0&0&1&|&0\\0&0&0&|&0\end{pmatrix}$$

Thus, by using back-substitution, we get $x_1-x_2=0,\ \ x_3=0$. Hence, the solutions are $x_1=x_2=t,\ x_3=0$. Therefore, the system has infinitely many one parametric solutions given by $S=\{(x_1,x_2,x_3)\}=\{(1,1,0)t : t\in R\}$.

$$e)\begin{pmatrix}1&-2&1&-1&|&2\\1&1&1&-1&|&2\\1&0&1&-1&|&2\end{pmatrix}\xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-R_1}\begin{pmatrix}1&-2&1&-1&|&2\\0&3&0&0&|&0\\0&2&0&0&|&0\end{pmatrix}$$

$$\xrightarrow{R_3\to 3R_3-2R_2}\begin{pmatrix}1&-2&1&-1&|&2\\0&3&0&0&|&0\\0&0&0&0&|&0\end{pmatrix}\xrightarrow[R_2\to\frac{1}{3}R_2]{R_1\to R_1+\frac{2}{3}R_2}\begin{pmatrix}1&0&1&-1&|&2\\0&1&0&0&|&0\\0&0&0&0&|&0\end{pmatrix}$$

Now, using $UX=C$, we get the new system $\begin{cases}x_1+x_3-x_4=2\\x_2=0\end{cases}$

Hence, by letting $x_3=r, x_4=t$, we get $x_1=t-r+2$ (two parametric solutions)

$$f)\begin{pmatrix}1&-2&1&-1&|&3\\1&1&1&-1&|&1\\1&0&1&-1&|&2\end{pmatrix}\xrightarrow[R_3\to R_3-R_1]{R_2\to R_2-R_1}\begin{pmatrix}1&-2&1&-1&|&3\\0&3&0&0&|&-2\\0&2&0&0&|&-1\end{pmatrix}$$

$$\xrightarrow{R_3\to 3R_3-2R_2}\begin{pmatrix}1&-2&1&-1&|&3\\0&3&0&0&|&-2\\0&0&0&0&|&1\end{pmatrix}\xrightarrow[R_1\to R_1+2R_2-3R_3]{R_2\to R_2+2R_3,\ R_2\to\frac{1}{3}R_2}\begin{pmatrix}1&0&1&-1&|&0\\0&1&0&0&|&0\\0&0&0&0&|&1\end{pmatrix}$$

Here, back substitution gives $0=1$ which is inconsistent. There is no solution.

D) Cramer's Rules

This method is applicable only when the system is square system (a square system means a system in which the number of equations is equal to the number of variables) and the coefficient matrix is non-singular.

For three equations and three unknowns:

Consider the system $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$

Let $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} k_1 & a_{12} & a_{13} \\ k_2 & a_{22} & a_{23} \\ k_3 & a_{32} & a_{33} \end{vmatrix}$

$\Delta_2 = \begin{vmatrix} a_{11} & k_1 & a_{13} \\ a_{21} & k_2 & a_{23} \\ a_{31} & k_3 & a_{33} \end{vmatrix}$, $\Delta_3 = \begin{vmatrix} a_{11} & a_{12} & k_1 \\ a_{21} & a_{22} & k_2 \\ a_{31} & a_{32} & k_3 \end{vmatrix}$

Therefore the solutions are $x_1 = \frac{\Delta_1}{\Delta}$, $x_2 = \frac{\Delta_2}{\Delta}$, $x_3 = \frac{\Delta_3}{\Delta}$.

Examples:

1. Solve the following systems using Crammer's rule

$a)\ \begin{cases} x_1 \ + x_3 = 0 \\ x_2 + x_3 = 5 \\ x_1 + x_2 = -1 \end{cases}$ $\qquad b)\ \begin{cases} 3x_1 - 2x_2 + 3x_3 = 0 \\ x_1 + 3x_2 - x_3 = -2 \\ x_1 + x_3 = 0 \end{cases}$

Solution:

$a)\ \Delta = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1-1 = -2,\ \Delta_1 = \begin{vmatrix} 0 & 0 & 1 \\ 5 & 1 & 1 \\ -1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ -1 & 1 \end{vmatrix} = 6$

$\Delta_2 = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 5 & 1 \\ 1 & -1 & 0 \end{vmatrix} = \begin{vmatrix} 5 & 1 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 5 \\ 1 & -1 \end{vmatrix} = -4,\ \Delta_3 = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 1 & -1 \end{vmatrix} = -6$

Therefore, $x_1=\frac{\Delta_1}{\Delta}=\frac{6}{-2}=-3, x_2=\frac{\Delta_2}{\Delta}=\frac{-4}{-2}=2, x_3=\frac{\Delta_3}{\Delta}=\frac{-6}{-2}=3$

b) $\Delta=\begin{vmatrix}3&-2&3\\1&3&-1\\1&0&1\end{vmatrix}=4,\quad \Delta_1=\begin{vmatrix}0&-2&3\\-2&3&-1\\0&0&1\end{vmatrix}=-4,\quad \Delta_2=\begin{vmatrix}3&0&3\\1&-2&-1\\1&0&1\end{vmatrix}=0$

$\Delta_3=\begin{vmatrix}3&-2&0\\1&3&-2\\1&0&0\end{vmatrix}=3\begin{vmatrix}3&-2\\0&0\end{vmatrix}+2\begin{vmatrix}1&-2\\1&0\end{vmatrix}=4$

Thus, $x_1=\frac{\Delta_1}{\Delta}=\frac{-4}{4}=-1,\ x_2=\frac{\Delta_2}{\Delta}=\frac{0}{4}=0,\ x_3=\frac{\Delta_3}{\Delta}=\frac{4}{4}=1$

Remarks (Cramer's rule as test of solutions):

i) If $\Delta\neq 0$, then the system is **consistent** and has a unique solution.

ii) If $\Delta=0$ and $\Delta_i=0$ for all $i=1,2,3,...n$, then the system is consistent and has infinitely many solutions (but it cannot be solved by Cramer's rule).

iii) If $\Delta=0$ and at least one of $\Delta_i\neq 0$, then the system has no solution.

Examples:

1. Using Crammer's rule, find the values of a and b for which the following system has a unique, many or no solutions.

i) $\begin{cases}x_1+ax_2=3\\x_1-2x_2=b\end{cases}$ ii) $\begin{cases}x+2y+3z=6\\x+3y+5z=9\\2x+5y+az=b\end{cases}$ iii) $\begin{cases}x+y+z=1\\x+2y+az=b\\x+4y+10z=b^2\end{cases}$

Solution:

i) In this system, $\Delta=\begin{vmatrix}1&a\\1&-2\end{vmatrix}=-2-a,\ \Delta_2=\begin{vmatrix}1&3\\1&b\end{vmatrix}=b-3$

A unique solution: The system will have a unique solution if $\Delta\neq 0\Rightarrow -2-a\neq 0\Rightarrow a\neq -2$.

So, the system has unique solution if $a\neq -2$ and $b\in R$.

Infinitely many solutions: The system will have infinitely many solutions if $\Delta=0, \Delta_i=0$ for all $i=1,2$. So, $\Delta=0\Rightarrow -2-a=0\Rightarrow a=-2$ and $\Delta_2=0\Rightarrow b-3=0\Rightarrow b=3$.

No solution: The system will have no solution if $\Delta=0$ and $\Delta_i \neq 0$ for at least one i. So, $\Delta=0 \Rightarrow -2-a=0 \Rightarrow a=-2$ and $\Delta_2 \neq 0 \Rightarrow b \neq 3$

ii) Here, $\Delta=\begin{vmatrix}1 & 2 & 3\\ 1 & 3 & 5\\ 2 & 5 & a\end{vmatrix}=\begin{vmatrix}3 & 5\\ 5 & a\end{vmatrix}-2\begin{vmatrix}1 & 5\\ 2 & a\end{vmatrix}+3\begin{vmatrix}1 & 3\\ 2 & 5\end{vmatrix}=a-8,$

$$\Delta_1=\begin{vmatrix}6 & 2 & 3\\ 9 & 3 & 5\\ b & 5 & 8\end{vmatrix}=6\begin{vmatrix}3 & 5\\ 5 & 8\end{vmatrix}-2\begin{vmatrix}9 & 5\\ b & 8\end{vmatrix}+3\begin{vmatrix}9 & 3\\ b & 5\end{vmatrix}=b-15$$

A unique solution: The system will have a unique solution if and only if $\Delta \neq 0 \Rightarrow a-8 \neq 0 \Rightarrow a \neq 8$. There is unique solution if $a \neq 8$ and $b \in R$.

Infinitely many solutions: The system will have infinitely many solutions if $\Delta=0$ and $\Delta_i=0$ for all $i=1,2,3$. So, $\Delta=0 \Rightarrow a-8=0 \Rightarrow a=8$ and $\Delta_1=0 \Rightarrow b-15=0 \Rightarrow b=15$.

Note that if we use Δ_2 and Δ_3, we get the same values for the constants.

No solution: The system will have no solution if $\Delta=0$ and $\Delta_i \neq 0$ for at least one i. So, $\Delta=0 \Rightarrow a-8=0 \Rightarrow a=8$ and $\Delta_1 \neq 0 \Rightarrow b-15 \neq 0 \Rightarrow b \neq 15$

iii) Similarly, $\Delta=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & a\\ 1 & 4 & 10\end{vmatrix}=\begin{vmatrix}2 & a\\ 4 & 10\end{vmatrix}-\begin{vmatrix}1 & a\\ 1 & 10\end{vmatrix}+\begin{vmatrix}1 & 2\\ 1 & 4\end{vmatrix}=-3a+12$

$$\Delta_3=\begin{vmatrix}1 & 1 & 1\\ 1 & 2 & b\\ 1 & 4 & b^2\end{vmatrix}=\begin{vmatrix}2 & b\\ 4 & b^2\end{vmatrix}-\begin{vmatrix}1 & b\\ 1 & b^2\end{vmatrix}+\begin{vmatrix}1 & 2\\ 1 & 4\end{vmatrix}=b^2-3b+2$$

A unique solution: The system will have a unique solution if $\Delta \neq 0 \Rightarrow -3a+12 \neq 0 \Rightarrow a \neq 4$ and $b \in R$.

Infinitely many solutions: The system will have infinitely many solutions if $\Delta=0$ and $\Delta_i=0$ for all $i=1,2,3$. So, $\Delta=0 \Rightarrow a=4$ and $\Delta_3=b^2-3b+2=0 \Rightarrow b=1, 2$.

No solution: The system will have no solution if $\Delta=0$ and $\Delta_i \neq 0$ for at least one i. So, $\Delta=0 \Rightarrow a=4$ and $\Delta_3 \neq 0 \Rightarrow b^2-3b+2 \neq 0 \Rightarrow b \neq 1, 2$.

1.8 Eigen-Values and Eigen-Vectors of Matrices

Definition: Suppose A is an $n\times n$ matrix. Then, a non-zero vector v in R^n is said to be an eigenvector of A if and only if $Av=\lambda v$ for some scalar λ. The scalar λ with this property is called an eigenvalue of A and the vector v is called an eigenvector corresponding to λ. Now, let v be an eigenvector corresponding to λ. Then, $Av=\lambda v \Rightarrow (A-\lambda I)v=0$, where I is identity matrix of the same size as A. The zero vector is the trivial solution for this equation. So, in order to have the equation $(A-\lambda I)v=0$ a non-trivial solution (more than one solution), the co-efficient matrix $A-\lambda I$ must be singular. That is $|A-\lambda I|=0$. This is called characteristics equation of A

Definition: Expanding the equation $|A-\lambda I|=0$, we get the polynomial in λ of $p(\lambda)=|A-\lambda I|$. This polynomial is called characteristics polynomial of A and the roots of the characteristics polynomial of A is called eigenvalues of A. The set of all eigenvectors corresponding to an eigenvalue λ is given by $E_\lambda=\{v:(A-\lambda I)v=0,\ v\neq 0\}$. The set of all eigen vectors corresponding to an eigen value λ together with the zero vector is known *eigen space.*

Remark: In order to find the eigenvalues of any square matrix A, simply find the roots of the characteristics equation of A and for each root λ compute the corresponding eigen vector v.

Examples:

1. Find the eigenvalues and the set of all the corresponding eigenvectors of the following matrices

$$a)\ A=\begin{pmatrix}3 & 2\\ 1 & 4\end{pmatrix}\quad b)\ A=\begin{pmatrix}3 & 1\\ 6 & 2\end{pmatrix}\quad c)\ A=\begin{pmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{pmatrix}$$

Solution:

$$a)\ |A-\lambda I|=0\Rightarrow\begin{vmatrix}3-\lambda & 2\\ 1 & 4-\lambda\end{vmatrix}=0\Rightarrow\lambda^2-7\lambda+10=0\Rightarrow\lambda=2,5$$

Hence, the eigenvalues are $\lambda=2$ and $\lambda=5$. Now let's find the eigenvectors.

i) For $\lambda = 2$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+2y=0 \\ x+2y=0 \end{cases} \Rightarrow x = -2y$

Hence, $E_2 = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : x = -2y, y \in R/\{0\}\right\} = \left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix} y : y \in R/\{0\}\right\}$

ii) For $\lambda = 5$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2x+2y=0 \\ x-y=0 \end{cases} \Rightarrow x = y$

Hence, $E_5 = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : x = y, y \in R/\{0\}\right\} = \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix} y : y \in R/\{0\}\right\}$.

b) $|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 1 \\ 6 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(2-\lambda) - 6 = 0$

$$\Rightarrow \lambda^2 - 5\lambda = 0 \Rightarrow \lambda(\lambda-5) = 0 \Rightarrow \lambda = 0, \lambda = 5$$

Hence, the eigenvalues are $\lambda = 0$ and $\lambda = 5$. Now let's find the eigen-vectors.

i) For $\lambda = 0$, $Av = \lambda v \Rightarrow \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3x+y=0 \\ 6x+2y=0 \end{cases} \Rightarrow y = -3x$

Hence, $E_0 = \left\{\begin{pmatrix} x \\ y \end{pmatrix} : y = -3x, x \in R/\{0\}\right\} = \left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix} x : x \in R/\{0\}\right\}$

ii) For $\lambda = 5$, $Av = \lambda v \Rightarrow \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 5\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{cases} 3x+y=5x \\ 6x+2y=5y \end{cases} \Rightarrow y = 2x$

c) $\begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)(\lambda^2 - 4\lambda + 3) = 0 \Rightarrow \lambda = 1,2,3$

Now let's find the eigenvectors corresponding to each of theses eigenvalues.

i) For $\lambda = 1$, $(A-\lambda I)v = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x+z=0 \\ y=0 \\ x+z=0 \end{cases} \Rightarrow x = -z,\ y = 0$

Hence, $E_1 = \left\{\begin{pmatrix} x \\ 0 \\ z \end{pmatrix} : x = -z, z \in R/\{0\}\right\} = \left\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} z : z \in R/\{0\}\right\}$

Hence, $E_2 = \left\{ \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} : y \in R/\{0\} \right\} = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} y : y \in R/\{0\} \right\}$

iii) For $\lambda = 3$, $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -x+z=0 \\ -y=0 \\ x-z=0 \end{cases} \Rightarrow x = z,\ y = 0$

Hence, $E_3 = \left\{ \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} : x = z, z \in R/\{0\} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} z : z \in R/\{0\} \right\}$

2. Find the eigen values and eigen vectors of $A = \begin{pmatrix} 6 & -3 \\ -8 & 4 \end{pmatrix}$.

Solution: First, find the eigen values

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 6-\lambda & -3 \\ -8 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (6-\lambda)(4-\lambda) - 24 = 0$$

$$\Rightarrow \lambda^2 - 10\lambda = 0 \Rightarrow \lambda = 0, \lambda = 10$$

Eigen-Vectors:

i) For $\lambda = 0$, $A\mathbf{v} = \lambda\mathbf{v} \Rightarrow \begin{pmatrix} 6 & -3 \\ -8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 6x - 3y = 0 \\ -8x + 4y = 0 \end{cases} \Rightarrow y = 2x$

Hence, $E_0 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = 2x, x \in R/\{0\} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} x : x \in R/\{0\} \right\}$

ii) For $\lambda = 10$, $\begin{pmatrix} 6 & -3 \\ -8 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10x \\ 10y \end{pmatrix} \Rightarrow \begin{cases} 6x - 3y = 10x \\ -8x + 4y = 10y \end{cases} \Rightarrow y = -\frac{4}{3}x$

Hence, $E_{10} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : y = -\frac{4}{3}x, x \in R/\{0\} \right\} = \left\{ \begin{pmatrix} 1 \\ -\frac{4}{3} \end{pmatrix} x : x \in R/\{0\} \right\}$

3. If $\lambda = 6$ is an eigenvalue of $A = \begin{pmatrix} 1 & -2 \\ k & 4 \end{pmatrix}$, then find k.

Solution: $|A - \lambda I| = 0 \Rightarrow |A - 6I| = 0 \Rightarrow \begin{vmatrix} -5 & -2 \\ k & -2 \end{vmatrix} = 0 \Rightarrow 2k + 10 = 0 \Rightarrow k = -5$

Eigenvalues with Multiplicity: Suppose $p(x)$ is the characteristics polynomial of an $n \times n$ matrix A. Then, we say that λ is an eigenvalue of A with multiplicity k if k is the largest positive integer for which $(x - \lambda)^k$ is a factor of $p(x)$. For instance, for $A = \begin{pmatrix} 5 & 3 & 7 \\ 0 & 4 & 0 \\ 0 & 6 & 5 \end{pmatrix}$ the characteristic polynomial is $p(\lambda) = (\lambda - 4)(\lambda - 5)^2$. Hence, the eigenvalues are $\lambda = 4, \lambda = 5$. Here, the largest integer for which $(\lambda - 4)^k$ is a factor of the polynomial $p(\lambda) = (\lambda - 4)(\lambda - 5)^2$ is $k = 1$ and the largest integer for which $(\lambda - 5)^k$ is a factor of the polynomial $p(\lambda) = (\lambda - 4)(\lambda - 5)^2$ is $k = 2$. Therefore, $\lambda = 4$ and $\lambda = 5$ are eigenvalues of A with multiplicity $k = 1$ and $k = 2$ respectively. In what follows, let's see how to find the eigenvectors of a matrix corresponding to eigenvalues with multiplicity of greater than one.

Examples: . Find eigenvalues and the corresponding eigenvectors of:

$a)\ A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$ $\quad b)\ A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$ $\quad c)\ A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

Solution:

$a)\ |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 2\lambda + 1 = 0 \Rightarrow (\lambda - 1)^2 = 0 \Rightarrow \lambda = 1$

Hence, the eigenvalue is $\lambda = 1$ with multiplicity 2. Now, let's find the corresponding eigenvector.

For $\lambda = 1$, $(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} y - x = 0 \\ y - x = 0 \end{cases} \Rightarrow x = y$.

Hence, $E_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x = y, y \in R/\{0\} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} t : t \in R/\{0\} \right\}$

b) $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0 \Rightarrow (\lambda - 5)(\lambda + 3)^2 = 0 \Rightarrow \lambda = 5, -3$

Hence, the eigenvalues are $\lambda = 5$ and $\lambda = -3$ with multiplicity 2.

Now let's find the eigenvectors corresponding to each of theses eigenvalues.

i) For $\lambda = 5$, $\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -7x + 2y - 3z = 0 \\ 2x - 4y - 6z = 0 \\ -x - 2y - 5z = 0 \end{cases}$

Here, adding the first and the third equations gives $z = -x$ and putting this in one of the equation gives again $y = 2x$

Hence, $E_5 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y = 2x,\ z = -x,\ x \in R/\{0\} \right\} = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} x : x \in R/\{0\} \right\}$

ii) For $\lambda = -3$, $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x + 2y - 3z = 0 \\ 2x + 4y - 6z = 0 \\ -x - 2y + 3z = 0 \end{cases}$

Now, finding the eigenvector satisfying the above system is equivalent to finding the non-trivial solution of the linear system $\begin{cases} x + 2y - 3z = 0 \\ 2x + 4y - 6z = 0 \\ -x - 2y + 3z = 0 \end{cases}$.

So, let's, solve this system by Gaussian elimination.

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus, by using back-substitution we have $x + 2y - 3z = 0$.

Then, letting $y = r, z = t$ and gives $x = 3t - 2r$.

Hence, $E_{-3}=\left\{\begin{pmatrix}3t-2r\\ r\\ t\end{pmatrix}: r,t\in R/\{0\}\right\}=\left\{\begin{pmatrix}3\\0\\1\end{pmatrix}t-\begin{pmatrix}2\\-1\\0\end{pmatrix}r: r,t\in R/\{0\}\right\}$

c) $|A-\lambda I|=0\Rightarrow\begin{vmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 0\\ -2 & 0 & 1-\lambda\end{vmatrix}=0\Rightarrow(1-\lambda)^3\Rightarrow\lambda=1$

In this case, we have got only one eigenvalue $\lambda=1$ with multiplicity of 3.

For $\lambda=1$, $\begin{pmatrix}0 & 2 & 3\\ 3 & 0 & 0\\ -2 & 0 & 0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\Rightarrow\begin{cases}2y+3z=0\\ 3x=0\Rightarrow x=0,\ y=-\frac{3}{2}z\\ -2x=0\end{cases}$

Hence, $E_1=\left\{\begin{pmatrix}0\\ \frac{3}{2}\\ -1\end{pmatrix}t: t\in R/\{0\}\right\}=\left\{\begin{pmatrix}0\\3\\-2\end{pmatrix}\frac{t}{2}: t\in R/\{0\}\right\}$

Review Problems on Chapter-1

1. Let $Y=\begin{pmatrix}-2 & -2 & 6\\ 0 & 2 & -10\\ 8 & -6 & 4\end{pmatrix}$. Find X so that $2X+3Y$ is a scalar matrix whose diagonal or non-zero entry is 2. **Answer:** $X=\begin{pmatrix}4 & 3 & -9\\ 0 & -2 & 15\\ -12 & 9 & -5\end{pmatrix}$

2. Let $A=\begin{pmatrix}2 & -1 & -3\\ -1 & 2 & -4\end{pmatrix}$ and $B=\begin{pmatrix}1 & 2\\ 3 & 0\\ 2 & 4\end{pmatrix}$. Find C if $C-2A=(3B)'$.

   Answer: $C=\begin{pmatrix}7 & 7 & 0\\ 4 & 4 & 4\end{pmatrix}$

3. If A has m rows and $m+5$ columns and B has n rows and $11-n$ columns, find m and n so that both products AB and BA exist. **Answer:** $m=3, n=8$

4. Given the matrices $F=\begin{pmatrix}2 & -3 & w\\ 1 & 5 & -2\end{pmatrix}, G=\begin{pmatrix}8 & x & -6\\ 7 & -2 & 1\\ 4 & y & -5\end{pmatrix}, P=\begin{pmatrix}7 & 23 & z\\ q & -12 & 9\end{pmatrix}$. If $FG=P$, then find q, x, y, z, w. **Answer:** $q=35, x=4, y=3, z=-30, w=3$

5. Given $A=\begin{pmatrix}7 & x\\ y & 3\\ 6 & -3\end{pmatrix}$ and $B=\begin{pmatrix}0 & a & 3\\ b & 4 & 4\\ 5 & c & 6\end{pmatrix}$. In $C=A'B$ if $c_{23}=9$, find x.

   Answer: $x=5$

6. If the ad joint of a matrix is $\begin{pmatrix}2 & -3\\ 4 & 5\end{pmatrix}$, find the matrix. **Answer:** $\begin{pmatrix}5 & 3\\ -4 & 2\end{pmatrix}$

7. If A and B are 2×2 matrices with $\det(A)=3$, $\det(B)=2$, find $\det(3AB')$.

   Answer : 54

8. If A, B and C are 2×2 with $\det A=-4$ and $C=(3B)^{-1}$, find $\det(6A^2B'C)$

9. Let A be 3×3 scalar matrix with $\det(3A)=216$. Then, find matrix A.

$$\textbf{Answer: } A=\begin{pmatrix}2&0&0\\0&2&0\\0&0&2\end{pmatrix}$$

10. Let A and B be 2×2 matrices with $\det A=2, \det B=3$. Find $\det(3A^2B^t)$.

Answer : 108

11. If A and B are 2×2 where $A=6A^{-1}$. Find $\det[B^tA^2(2B)^{-1}]$. **Answer** : 9

12*. Let $X=\begin{pmatrix}-8&3&6\\0&-2&9\\3&12&-5\end{pmatrix}$. Find a matrix Y so that $3Y-2X$ is a scalar matrix with $\det(4X-6Y)=64$.

$$\textbf{Answer: } Y=\begin{pmatrix}7&2&4\\0&-2&6\\2&8&-4\end{pmatrix}$$

13. Suppose A and B are 2×2 non-singular matrices such that $A=5B^{-1}$. Then, find $\det(2AB^2A^t)$. **Answer** : 2500

14. If A is a 3×3 matrix with $\det A=5$, find $\det[2(\text{adj } A)]$. **Answer**: 200

15. Let A, B be 2×2 matrices such that, $\det(B)=4$. Then, find $\det(3AB^tA^{-1})$.

Answer : 36

16*. If A, B and C are 3×3 matrices where $\det A=16$ and $\det B=6$, then, find $\det[A^{-2}adj(2B)^t]$ and $\det[3(2AC)^{-1}(4BC)^t]$. **Answer**: 18 & 81

17*. Given $A=\begin{pmatrix}1&-1&2&3\\2&1&h&1\\0&-3&k&5\\3&3&4&-1\end{pmatrix}$. Find the values of h and k if $rank(A)=2$.

Answer : $h=3, k=1$

18. Let A, B and C be 3×3 with $\det A=3, \det(ABC)=96$. If $C=2B$, then find $\det(B)$ and $\det(C)$. **Answer** : $\det B=\pm2$, $\det C=\pm16$

19. Suppose $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$. Then, find

a) $\begin{vmatrix} 6a & 3b & 15c \\ 2d & e & 5f \\ 8g & 4h & 20i \end{vmatrix}$ b) $\begin{vmatrix} 3d-2a & d & g \\ 9f-6c & 3f & 3i \\ 3e-2b & e & h \end{vmatrix}$ c) $\begin{vmatrix} 15a-7b & 3b & 6c \\ 5d-7e & e & 2f \\ 20g-7h & 4h & 8i \end{vmatrix}$

Answer: *a)* 600 *b)* 30 *c)* 600

20. Suppose A, B and C are 3×3 matrices with $B = 2A^{-1}$ and $\det(ABC) = 32$. Then, find $\det(2AB'C^{-1})$.

21. Find the inverse of each of the following matrices.

a) $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ b) $A = \begin{pmatrix} -1 & 3 & 0 \\ 3 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ -1 & 2 & 4 \end{pmatrix}$

Answer: *a)* $\begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$ *b)* $\begin{pmatrix} 5 & 6 & -3 \\ 2 & 2 & -1 \\ -1 & -1 & 1 \end{pmatrix}$ *c)* $\begin{pmatrix} -2 & 2 & -1 \\ 5 & -3 & 2 \\ -3 & 2 & -1 \end{pmatrix}$

22*. Suppose the ad-joint of A is $\begin{pmatrix} 10 & 16 & -12 \\ 5 & 10 & -6 \\ -6 & -12 & 8 \end{pmatrix}$. If $\det A < 0$, find A^{-1}.

[**Hint:** Do not try to find A rather use short-cut and the condition $\det A < 0$]

23. Find the values of a, k (if any) so that the systems have i) unique solution ii) many solutions iii) no solution

a) $\begin{cases} x_1 - x_2 = 2 \\ x_1 + kx_2 = 1 \end{cases}$ b) $\begin{cases} kx + y + z = 1 \\ x + ky + z = 1 \\ x + y + kz = 1 \end{cases}$ c) $\begin{cases} x + 2y + 3z = 0 \\ kx + y + 2z = 0 \\ 6x + y + kz = 0 \end{cases}$

Answer: a) *i)* $k \neq -1$ *ii)* None *iii)* $k = -1$

b) *i)* $k \neq 1, -2$ *ii)* $k = 1$ *iii)* $k = -2$

c) *i)* $k \neq 5, -8$ *ii)* $k = 5, -8$ *iii)* None

24. Solve the following systems using Gaussian-Elimination method.

$a)\begin{cases} x_1+x_2+x_3+x_4=1 \\ x_1+2x_2+2x_3+2x_4=1 \\ x_1+2x_2+3x_3+3x_4=1 \end{cases}$ $b)\begin{cases} x_1-x_2+2x_3+x_4=2 \\ 2x_1-3x_2+3x_3+2x_4=5 \\ 3x_1-3x_2+6x_3+4x_4=-1 \end{cases}$ $c)\begin{cases} x_1-2x_2+x_3-x_4=3 \\ 2x_1-4x_2+x_3+x_4=2 \\ x_1-2x_2-2x_3+3x_4=1 \end{cases}$

Answer : $a)\{(1,0,-t,t):t\in R\}$ $b)\{(8-3t,-1-t,t,-7):t\in R\}$ $c)\{(2t+3,t,-2,-2):t\in R\}$

25. Using Gauss-Jordan Method, solve the following systems.

$a)\begin{cases} 2x_1+x_2-3x_3-x_4=2 \\ x_1-x_2-3x_3+x_4=4 \\ -6x_1-3x_2+9x_3+3x_4=-6 \end{cases}$ $b)\begin{cases} 2x_1+x_2+2x_3+3x_4=5 \\ 6x_1+2x_2+4x_3+8x_4=14 \\ x_1-x_2+4x_4=5 \\ x_2+2x_3-4x_4=-4 \end{cases}$

Answer : $a)x_4=r, x_3=t, x_2=r-t-2, x_1=2t-r+2$ $b)\{(1,0,0,1)\}$

26*. Using Cramer's rule, solve the following system for α, β, γ where

$-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}, 0\le\beta\le\pi$ and $-\frac{\pi}{2}\le\gamma\le\frac{\pi}{2}$ $\begin{cases} 2\sin\alpha-\cos\beta+3\tan\gamma=3 \\ 4\sin\alpha+2\cos\beta-2\tan\gamma=2 \\ 6\sin\alpha-3\cos\beta+\tan\gamma=9 \end{cases}$

Answer: $\alpha=\frac{\pi}{2}, \beta=\pi, \gamma=0$

27*. Suppose $\det A=-1$ and *cofactor matrix of A is* $C=\begin{pmatrix} -1 & 1 & 0 \\ -2 & 0 & -3 \\ -3 & 0 & -4 \end{pmatrix}$.

a) Find A^{-1} b) Solve the system $AX=B'$ where $B=(1\ \ 0\ \ 2)$

28. Find the eigen values and the corresponding eigen vectors.

$a)A=\begin{pmatrix} 6 & 16 \\ -1 & -4 \end{pmatrix}$ $b)A=\begin{pmatrix} 2 & 3 \\ 4 & 6 \end{pmatrix}$ $c)A=\begin{pmatrix} 3 & 0 & -1 \\ 2 & 3 & 2 \\ -1 & 0 & 3 \end{pmatrix}$ $d)A=\begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{pmatrix}$

Answer : $a)-2,4;\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}r:r\in R/\{0\}\right\},\left\{\begin{pmatrix} -8 \\ 1 \end{pmatrix}r:r\in R/\{0\}\right\}$

$b)\ \lambda=0,8;\left\{\begin{pmatrix} 3 \\ -2 \end{pmatrix}r\right\},\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}r:r\in R/\{0\}\right\}$ $c)2,3,4$ $d)\ 1,2$

CHAPTER-2

ANALYSIS OF VECTORS

2.1 Introduction to Vectors

An ordered n-tuple of numbers $(a_1, a_2, ..., a_n)$ is called a vector. Here, $a_1, a_2, a_3, ..., a_n$ are called components. The set of all vectors with n-components is denoted by R^n.

Notation: Vectors are denoted by small case letter with an arrow as $\vec{a}, \vec{b}, \vec{c}$.

In a pane R^2: Vectors are of the form $\vec{a} = (a_1, a_2)$.

In space R^3: Vectors are of the form $\vec{a} = (a_1, a_2, a_3)$.

Zero vector: Suppose $\vec{a} = (a_1, a_2, ..., a_n)$ is any vector in R^n. Then, $\vec{a}$ is said to be zero vector if and only if all the components are zero.

That is $\vec{a} = 0 \Leftrightarrow a_i = 0, \forall i = 1,2,3,...,n$.

Position Vector: It is a directed vector $\vec{r} = \overline{OA} = A - O = A$ formed between the origin O and the position of a point A. From this concept, we can say that any point in a plane represents some position vector.

Directed Vector: Any vector with initial point A and terminal point B denoted by $\overline{AB}$ is called directed vector from A to B. It is obtained by $\overline{AB} = B - A$.

Equal Vectors: Any two vectors are said to be equal if and only if their corresponding components are all equal.

Particularly, for $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases}$

2.2 Operations on Vectors

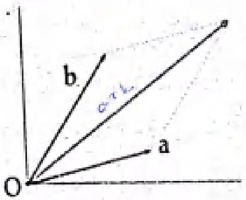
Suppose $\vec{a}=(a_1,a_2,a_3)$ and $\vec{b}=(b_1,b_2,b_3)$ are any two vectors. Then, we define the sum and difference of these vectors as follows.

Addition of Vectors: $\vec{a}+\vec{b}=(a_1+b_1,a_2+b_2,a_3+b_3)$. That is, $\vec{a}+\vec{b}$, is a new vector obtained by adding the corresponding components of $\vec{a}$ and $\vec{b}$.

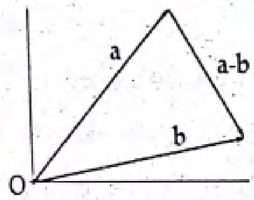
Subtraction of Vectors: $\vec{a}-\vec{b}=(a_1-b_1,a_2-b_2,a_3-b_3)$. That is, $\vec{a}-\vec{b}$, is a new vector obtained by subtracting the corresponding components of $\vec{a}$ and $\vec{b}$.

Scalar Multiplication of a vector: If k is a scalar, then $k\vec{a}$ is a vector given by $k\vec{a}=(ka_1,ka_2,ka_3)$.

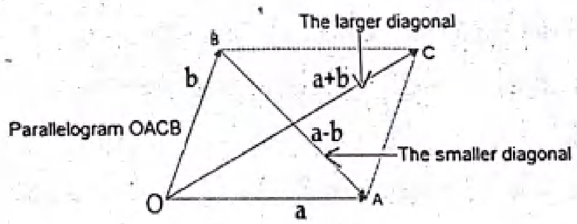
The parallelogram law: The vectors along the diagonals are sum and difference of the two vectors representing the sides of the parallelogram.



a) The sum of vectors



b) The difference of vectors



c) The Parallelogram Law

Examples:

1. Find x, y, z if $\vec{a} = (x-y, x+y, 7)$ and $\vec{b} = (5, -1, 3z+1)$ are equal.

Solution: Use equality of vectors: $\vec{a} = \vec{b} \Leftrightarrow a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$.

That is $\vec{a} = \vec{b} \Leftrightarrow \begin{cases} a_1 = b_1 \\ a_2 = b_2 \\ a_3 = b_3 \end{cases} \Leftrightarrow \begin{cases} x - y = 5 \\ x + y = -1 \\ 7 = 3z + 1 \end{cases} \Leftrightarrow x = 2, y = -3, z = 2$

2. Let $\vec{a} = (2,3,5)$ and $\vec{b} = (1,-1,3)$. Find $\vec{a}+\vec{b},\ \vec{a}-\vec{b}, 3\vec{a}, 2\vec{a}-3\vec{b}$

Solution: We use sum, difference and scalar multiple definitions.

Addition: $\vec{a}+\vec{b} = (a_1+b_1, a_2+b_2, a_3+b_3) = (2+1, 3-1, 5+3) = (3,2,8)$

Subtraction: $\vec{a}-\vec{b} = (a_1-b_1, a_2-b_2, a_3-b_3) = (2-1, 3+1, 5-3) = (1,4,2)$

Scalar multiplication: $3\vec{a} = (3a_1, 3a_2, 3a_3) = (3,9,15)$

Any combination: $2\vec{a}+3\vec{b} = (2a_1+3b_1, 2a_2+3b_2, 2a_3+3b_3) = (7,3,19)$

3. Find a vector $\vec{v}$ directed from A to B where $A = (2,-1,3)$ and $B = (5,-4,7)$.

Solution: By definition of directed vector: $\vec{v} = \overrightarrow{AB} = B - A = (3,-3,4)$.

4. Let $\overrightarrow{AB} = (-2,0,-4)$. If the midpoint of the segment $\overline{AB}$ is $(2,3,-2)$, find the end points A and B.

Solution: Let $A = (a,b,c)$ and $B = (x,y,z)$. Then, $\overrightarrow{AB} = (x-a, y-b, z-c)$.

Now, use definition of equality on the given value $\overrightarrow{AB} = (-2,0,-4)$.

That is $\overrightarrow{AB} = (x-a, y-b, z-c) = (-2,0,-4) \Rightarrow x-a=-2, y-b=0, z-c=-4$.

On the other hand, use the definition of the mid-point of $\overline{AB}$ as follow:

$$M = \frac{1}{2}(A+B) = \frac{1}{2}(a+x, b+y, c+z) = \left(\frac{a+x}{2}, \frac{b+y}{2}, \frac{c+z}{2}\right) = (2,3,-2)$$

$$\Rightarrow (a+x, b+y, c+z) = (4,6,-4)$$

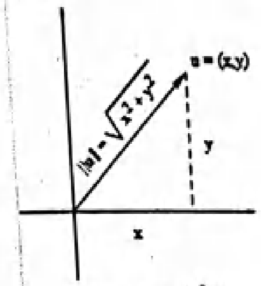
$$\Rightarrow a+x=4, b+y=6, c+z=-4$$

$$\Rightarrow 2a=6, 2b=6, 2c=0 \Rightarrow a=3, b=3, c=0$$

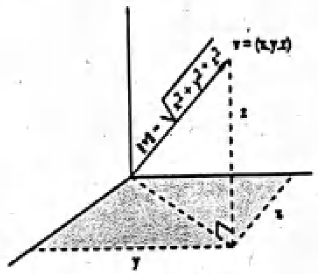
Therefore, $A = (a,b,c) = (3,3,0)$, $B = (x,y,z) = (1,3,-4)$

2.3 Norm of a Vector and the concept of Unit Vectors

Let $\vec{a} = (a_1, a_2, a_3)$ be a vector. Then, the norm or magnitude of $\vec{a}$ is the quantity defined by $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$. Geometrically, the norms of $\vec{v} = (x, y)$ in R^2 and $\vec{v} = (x, y, z)$ in R^3 represent the distances (lengths) of the line segments from the origin to the points (x, y) and (x, y, z) as shown below.



i) Norm in R^2



ii) Norm in R^3

Unit-Vector and Standard Unit-Vectors:

Unit vector: A vector having norm of 1 (a norm of unity) is called *unit vector*. That is a vector **u** is a unit vector if and only if $\|\mathbf{u}\| = 1$.

Standard unit vectors:

Let $\vec{a} = (a_1, a_2, a_3)$ be any vector in R^3. Then, we can express this vector as a sum of three vectors each with two components zero as follow:

$\vec{a} = (a_1, a_2, a_3) = (a_1, 0, 0) + (0, a_2, 0) + (0, 0, a_3) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$.

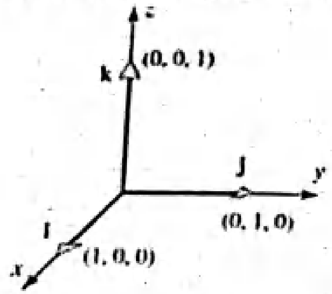
From this decomposition, we can say that any vector in R^3 is expressible as a combination of the three vectors: $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

So, if we denote these vectors by $i = (1,0,0)$, $j = (0,1,0)$ and $k = (0,0,1)$.

Then, vector $\vec{a}$ can be expressed as $\vec{a} = a_1 i + a_2 j + a_3 k$ with $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$.

Here, the unit vectors $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$ and $\mathbf{k} = (0,0,1)$ are called *standard (basic) unit vectors*.

Orientation of the standard unit vectors: Furthermore, these unit vectors are orthogonal and they are directed along the x-,y- and z-coordinate axes pointing in the positive directions as shown in the diagram below.



Orientation of the standard unit vectors

Notations: Still now we used the coordinate notation $\vec{a} = (a_1, a_2, a_3)$. From now onwards, we use the standard notation or **i-j-k** notation as : $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$. Every concept like operation and others that we discussed about vectors holds true with this notation.

Formula for Finding Unit vectors to a given vector:

Always, there are two unit vectors parallel to a non-zero vector $\vec{a}$. Of the two unit vectors, one of them is in the same direction whereas the other one is in opposite direction to $\vec{a}$. These unit vectors can be determined as follow:

In the same direction to $\vec{a}$: A unit vector in the same direction is $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|}$.

In opposite direction to $\vec{a}$: A unit vector in opposite direction is $\vec{u} = -\frac{\vec{a}}{\|\vec{a}\|}$.

Importance of unit vectors: Once a unit vector of a certain vector is known, it is possible to determine the vector itself if its norm is known or can be found.

Useful Properties of Norm: Suppose $\vec{a}$ and $\vec{b}$ are non-zero vectors.

a) $\|\vec{a}\| \geq 0$ & $\|\vec{a}\| = 0 \Leftrightarrow \vec{a} = 0$ *b)* $\|k\vec{a}\| = |k|\|\vec{a}\|$

c) $\|\vec{a}\|^2 = \vec{a}.\vec{a} = \vec{a}^2$ & $\|\vec{a}\| = \sqrt{\vec{a}.\vec{a}}$

Examples:

1. Find the norms of the vectors $\vec{a} = 4\mathbf{i} + \mathbf{j} + 8\mathbf{k}$ and $\vec{b} = (2,4,8,4)$.

Solution: Use the definition of norm: $\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$:

$\vec{a} = 4\mathbf{i} + \mathbf{j} + 8\mathbf{k} \Rightarrow \|\vec{a}\| = \sqrt{(4)^2 + (1)^2 + (8)^2} = \sqrt{16+1+64} = \sqrt{81} = 9$.

$\vec{b} = (2,4,8,4) \Rightarrow b = \sqrt{(2)^2 + (4)^2 + (8)^2 + (4)^2} = \sqrt{100} = 10$

2. For the vector $\vec{a} = (x,2,x)$ and $\|\vec{a}\| = 6$, find the values of x.

Solution: Here, $\|\vec{a}\| = \sqrt{x^2 + 4 + x^2} = \sqrt{2x^2 + 4}$.

Since norm of a vector is unique, equate with the given norm $\|\vec{a}\| = 6$.

That is $\|\vec{a}\| = 6 \Rightarrow \sqrt{2x^2 + 4} = 6 \Rightarrow 2x^2 + 4 = 36 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$.

3. Identify the unit vector from $\vec{u} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ and $\vec{u} = \frac{4}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{4}{7}\mathbf{k}$.

Solution: A vector $\mathbf{u}$ is a unit vector if and only if $\|\mathbf{u}\| = 1$.

For $\vec{u} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$, $\|\mathbf{u}\| = \sqrt{(\frac{1}{3})^2 + (\frac{2}{3})^2 + (\frac{2}{3})^2} = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{9}{9}} = \sqrt{1} = 1$.

So, $\vec{u} = \frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$ is a unit vector.

But $\vec{u} = \frac{4}{7}\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{4}{7}\mathbf{k}$ is not unit vector because $\|\mathbf{u}\| = \sqrt{(\frac{4}{7})^2 + (\frac{2}{7})^2 + (\frac{4}{7})^2} = \frac{6}{7} \neq 1$.

4. If $\vec{u} = \frac{6t}{13}\mathbf{i} + \frac{4}{13}\mathbf{j} + \frac{5}{13}\mathbf{k}$ is a unit vector, find the value of t.

Solution: Here, $\vec{u} = \frac{6t}{13}\mathbf{i} + \frac{4}{13}\mathbf{j} + \frac{5}{13}\mathbf{k} \Rightarrow \|\mathbf{u}\| = \sqrt{(\frac{6t}{13})^2 + (\frac{4}{13})^2 + (\frac{5}{13})^2} = \sqrt{\frac{36t^2}{169} + \frac{25}{169}}$.

So, $\|\mathbf{u}\| = 1 \Rightarrow \sqrt{\frac{36t^2}{169} + \frac{25}{169}} = 1 \Rightarrow \frac{36t^2}{169} + \frac{25}{169} = 1 \Rightarrow \frac{36t^2}{169} = \frac{144}{169} \Rightarrow t^2 = 4 \Rightarrow t = \pm 2$.

5. Let $\vec{a} = 2i - j + 2k$. Find a unit vector in the same and opposite direction of $\vec{a}$

Solution: Here, $\|\vec{a}\| = 3$. So, use the unit vector formula as follow:

In the same direction: $\vec{u} = \frac{\vec{a}}{\|\vec{a}\|} = \frac{1}{3}(2i - j + 2k) = \frac{2}{3}i - \frac{1}{3}j + \frac{2}{3}k$.

In opposite direction: $\vec{u}=-\frac{\vec{a}}{\|\vec{a}\|}=-\frac{1}{3}(2i-j+2k)=-\frac{2}{3}i+\frac{1}{3}j-\frac{2}{3}k$.

6. Given vector $\vec{a}$ where $\|\vec{a}\|=6$ and $\|k\vec{a}\|=4$. Find the scalar k.

Solution: By properties of norm, $\|k\vec{a}\|=|k|\|\vec{a}\| \Rightarrow 6|k|=4 \Rightarrow |k|=\frac{2}{3} \Rightarrow k=\pm\frac{2}{3}$

7. Given vector $\vec{a}$ where $\vec{a}.\vec{a}=8$. Find $\|\sqrt{2}\vec{a}\|$.

Solution: From properties of norm, $\|\sqrt{2}\vec{a}\|=\sqrt{2}\|\vec{a}\|$ and $\vec{a}.\vec{a}=\|\vec{a}\|^2$.

So, $\vec{a}.\vec{a}=8 \Rightarrow \|\vec{a}\|^2=8 \Rightarrow \|\vec{a}\|=\sqrt{8}=2\sqrt{2}$. So, $\|\sqrt{2}\vec{a}\|=\sqrt{2}\|\vec{a}\|=\sqrt{2}(2\sqrt{2})=4$.

8. Given $\vec{a}=(1,-5,6)$ and $\vec{b}=(3,y+8,-6)$. If $\|\vec{a}+\vec{b}\|=5$, find the value of y

Solution: Here, $\vec{a}+\vec{b}=(4,y+3,0)$. Then,

$$\|\vec{a}+\vec{b}\|=5 \Rightarrow \sqrt{16+(y+3)^2}=5 \Rightarrow \sqrt{y^2+6y+25}=5$$

$$\Rightarrow y^2+6y+25=25 \Rightarrow y^2+6y=0 \Rightarrow y=0, y=-6$$

9. Let $\vec{a}=i-j+k$, $\vec{b}=3i+5j+k$. Find a unit vector in the direction of $\vec{a}+\vec{b}$.

Solution: A unit vector in the direction of $\vec{a}+\vec{b}$ is given by $\vec{u}=\frac{\vec{a}+\vec{b}}{\|\vec{a}+\vec{b}\|}$ where

$\vec{a}+\vec{b}=4i+4j+2k$. So, $\vec{u}=\frac{1}{6}(4i+4j+2k)=\frac{2}{3}i+\frac{2}{3}j+\frac{1}{3}k$

2.4 Dot and Cross Products of Vectors

Suppose $\vec{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\vec{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ are any two vectors.

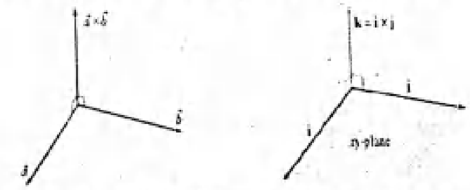
Dot (Scalar) Product: The *dot product* of $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}.\vec{b}$ is defined as $\vec{a}.\vec{b} = a_1.b_1 + a_2.b_2 + a_3.b_3$. The name dot product comes from the operation used.

Cross (Vector) Product: The *cross-product* of $\vec{a}$ and $\vec{b}$ denoted by $\vec{a}\times\vec{b}$ is defined as $\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$.

Short-form for cross product: The above component formula of cross product is represented using determinant (we will see determinant under matrix topic but for now, we used it because it simplifies cross product of vectors).

Cross product: $\vec{a}\times\vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Notice that the result of dot product is a scalar whereas the result of cross product is another vector. Geometrically, $\vec{a}\times\vec{b}$ is a vector perpendicular to both $\vec{a}$ and $\vec{b}$ as well as to the plane containing them as shown below.



Orientation of a×b with respect to a and b   Cross product using the standard unit vectors

Orthogonal (Perpendicular) and Parallel Vectors

Orthogonal Vectors: Two non-zero vectors $\vec{a}$ and $\vec{b}$ are said to be orthogonal if their dot product is zero. Denoted by $\vec{a}\perp\vec{b}$. Thus, $\vec{a}\perp\vec{b} \Leftrightarrow \vec{a}.\vec{b} = 0$.

Parallel Vectors: Any two non-zero vectors are said to be parallel if and only if one is the scalar multiple of the other. In other words, any two non-zero vectors are said to be parallel if and only if their cross product is zero.

That is $\vec{a} // \vec{b} \Leftrightarrow \vec{b} = t\vec{a}$ or $\vec{a} = t\vec{b}$ for some $t \in R/\{0\}$.

Examples: On dot and cross products and on perpendicular and parallel vectors.

1. Given the vectors $\vec{a} = 2i + j, \vec{b} = 4i + j + k$. Then,

   a) Find dot and cross products of the two vectors

   b) Give a vector perpendicular to both $\vec{a}$ and $\vec{b}$

   c) Give a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

   d) List at least three vectors perpendicular to $\vec{a}$

   e) Give a unit vector in the direction of $2\vec{a} \times \vec{b}$

   f) Find a vector with norm 9 in the same direction of $\vec{a} \times \vec{b}$

Solution:

a) Dot and cross products of two vectors:

Dot product: $\vec{a}.\vec{b} = a_1.b_1 + a_2.b_2 + a_3.b_3 = 8 + 1 + 0 = 9$

Cross product: $\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} i & j & k \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{vmatrix} = i - 2j - 2k$

b) **Finding perpendicular vectors:**

Always the vector $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$. Thus, the vector $\vec{a} \times \vec{b} = i - 2j - 2k$ and all its scalar multiples are perpendicular to $\vec{a}$ and $\vec{b}$.

c) **Finding perpendicular unit vector:**

Always a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is the unit vector $\vec{u} = \frac{\vec{a} \times \vec{b}}{\|\vec{a} \times \vec{b}\|}$.

But from part (a), $\vec{a} \times \vec{b} = i - 2j - 2k$ and thus $\|\vec{a} \times \vec{b}\| = 3$. Therefore, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\vec{u} = \frac{\vec{a} \times \vec{b}}{\|\vec{a} \times \vec{b}\|} = \frac{1}{3}(i - 2j - 2k)$.

d) Since $\vec{a} \times \vec{b}$ is perpendicular to $\vec{a}$, for any scalar t, the vector $t.(\vec{a} \times \vec{b})$ is also perpendicular to $\vec{a}$. So, we can produce many vectors like $2(\vec{a} \times \vec{b}) = 2i - 4j - 4k$ $3(\vec{a} \times \vec{b}) = 3i - 6j - 6k$ and as examples.

e) A unit vector in the direction of $2\vec{a}\times\vec{b}$ is $\vec{u}=\frac{2\vec{a}\times\vec{b}}{|2\vec{a}\times\vec{b}|}$. But from part (a),

$\vec{a}\times\vec{b}=i-2j-2k$ and thus $2\vec{a}\times\vec{b}=2(\vec{a}\times\vec{b})=2(i-2j-2k)=2i-4j-4k$.

Besides, $\|2\vec{a}\times\vec{b}\|=6$. Therefore, a unit vector in the direction of $2\vec{a}\times\vec{b}$ is

$\vec{u}=\frac{2\vec{a}\times\vec{b}}{|2\vec{a}\times\vec{b}|}=\frac{1}{6}(2i-4j-4k)=\frac{1}{3}(i-2j-2k)$.

f) A vector with norm 9 in the same direction of $\vec{a}\times\vec{b}$ is $\vec{c}=9\vec{u}$ where $\vec{u}$ a unit vector in the direction of $\vec{a}\times\vec{b}$. But we did in part (c), $\vec{u}=\frac{1}{3}(i-2j-2k)$

Therefore, $\vec{v}=9\vec{u}=\frac{9}{3}(i-2j-2k)=3i-6j-6k$.

2. Let $\vec{a}=3i+xj-5k$, $\vec{b}=2i-2j+4k$. Find the scalar x such that $\vec{a}.\vec{b}=-2$.

Solution: Here, $\vec{a}.\vec{b}=-2\Rightarrow 6-2x-20=-2\Rightarrow -2x=12\Rightarrow x=-6$.

3. Let $\vec{a}=(3,-t,2)$ and $\vec{b}=(5t,17,3)$. Find t so that $\vec{a}$ and $\vec{b}$ are orthogonal.

Solution: $\vec{a}\perp\vec{b}\Rightarrow\vec{a}\vec{b}=0\Rightarrow 15t-17t+6=0\Rightarrow -2t=-6\Rightarrow t=3$.

4. Let $\vec{a}=(m,3,-4)$ and $\vec{b}=(2,-n,8)$. Find m and n so that $\vec{a}$ and $\vec{b}$ are parallel.

Solution: Here, from the definition of parallel vectors,

$$\vec{a}//\vec{b}\Leftrightarrow\exists t\in R,\ni\vec{a}=t\vec{b}\Leftrightarrow(m,3,-4)=t(2,-n,8)\Leftrightarrow m=2t,3=-nt,-4=8t$$
$$\Leftrightarrow t=-1/2,m=-1,n=6$$

5. The dot products of a certain vector $\vec{v}$ with the vectors $\vec{a}=5k$, $\vec{b}=j$, $\vec{c}=3i$ are 20, 6 and 3 respectively, find the vector $\vec{v}$.

Solution: Let the required vector be $\vec{v}=ai+bj+ck$.

Then, $\vec{a}.\vec{v}=20\Rightarrow 5c=20\Rightarrow c=4$, $\vec{b}.\vec{v}=6\Rightarrow b=6$, $\vec{a}.\vec{v}=3\Rightarrow 3a=3\Rightarrow a=1$

Therefore, $\vec{v}=ai+bj+ck\Rightarrow\vec{v}=i+6j+4k$

6. Let $\vec{a}=2i-3j$, $\vec{b}=2i$. Find a vector $\vec{c}$ in a plane so that $\vec{c}.\vec{b}=6$ and $\vec{c}\perp\vec{a}$.

Solution: Here, let $\vec{c}=xi+yj$. Then, $\vec{c}.\vec{b}=6\Rightarrow 2x-0=6\Rightarrow x=3$.

Again, $\vec{c}\perp\vec{a}\Rightarrow\vec{c}.\vec{a}=0\Rightarrow 6-3y=0\Rightarrow y=2\Rightarrow\vec{c}=xi+yj=3i+2j$.

7. Given $A=(2,3,0)$, $C=(3,-4,1)$, $D=(5,-4,2)$ and $\|\overrightarrow{AB}\|=5$. If the vector $\overrightarrow{AB}$ is parallel the vector $\mathbf{v}=\overrightarrow{CD}$, find the possible coordinates of point B.

Solution: Let $B=(x,y,z)$. Then, $\overrightarrow{AB}=B-A=(x-2,y-3,z)$. Similarly, using the given coordinates, $\mathbf{v}=\overrightarrow{CD}=D-C=(2,0,1)$. But if two vectors are parallel, one is a scalar multiple of the other. So, $\overrightarrow{AB}\,//\,\mathbf{v}\Rightarrow\overrightarrow{AB}=t\mathbf{v}$ for some scalar t.

Using $\overrightarrow{AB}=(x-2,y-3,z)$ and $\mathbf{v}=(2,0,1)$, we can solve for B in terms of t

$$\overrightarrow{AB}=t\mathbf{v}\Rightarrow B-A=t(2,0,1)\Rightarrow(x-2,y-3,z)=t(2,0,1)$$
$$\Rightarrow x-2=2t,\ y-3=0,\ z=t\Rightarrow x=2t+2,\ y=3,\ z=t$$
$$\Rightarrow B=(2t+2,3,t)$$

So, $\overrightarrow{AB}=B-A=(x-2,y-3,z)=(2t,0,t)\Rightarrow\|\overrightarrow{AB}\|=\sqrt{(2t)^2+t^2}=\sqrt{4t^2}=2|t|$.

Besides, we are given the norm $\|\overrightarrow{AB}\|=4$. Since the norm of a vector is unique, we have $\|\overrightarrow{AB}\|=2|t|=4\Rightarrow|t|=2\Rightarrow t=\pm2$.

Therefore, the coordinates of B are determined as follow:

If $t=2$, then $B=(2t+2,3,t)\Rightarrow B=(6,3,2)$ and if $t=-2$, $B=(-2,3,-2)$.

8. Let $\vec{v}=ai+bj$. If $\|\vec{v}\|=\sqrt{8}$ and $\vec{v}$ is perpendicular to $u=-i+j$, then find $\vec{v}$.

Solution: Since $\vec{v}\perp\vec{u}$, we have $\vec{u}.\vec{v}=0\Rightarrow b-a=0\Rightarrow a=b$.

Besides, $\|\vec{v}\|=\sqrt{8}\Rightarrow a^2+b^2=8\Rightarrow 2a^2=8\Rightarrow a=\pm2$. Therefore, $\vec{v}=\pm2(i+j)$.

9. Let $v=i+j$. If $u=ai+aj+ck$ is a unit vector with $\|v-2u\|=\sqrt{2}$, find u.

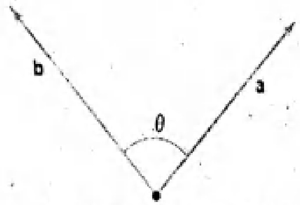
Solution: Here, $\|\vec{u}\|=\sqrt{2a^2+c^2}=1\Rightarrow\|\vec{u}\|^2=2a^2+c^2=1\Rightarrow c^2=1-2a^2$ and

$$\|\vec{v}-2\vec{u}\|=\sqrt{2}\Rightarrow\|\vec{v}-2\vec{u}\|^2=2\Rightarrow(1-2a)^2+(1-2a)^2+4c^2=2$$
$$\Rightarrow 2(4a^2-4a+1)+4(1-2a^2)=2$$
$$\Rightarrow -8a+6=2\Rightarrow a=\frac{1}{2},\ c=\pm\frac{\sqrt{2}}{2}\Rightarrow\vec{u}=\frac{1}{2}i+\frac{1}{2}j\pm\frac{\sqrt{2}}{2}k$$

2.5 Angle between Vectors and Projection of a Vector

Angle Between Two Vectors:

The angle between two non-zero vectors $\vec{a}$ and $\vec{b}$ is an angle θ with domain $0 \le \theta \le \pi$ when the vectors share (or made to share) the same initial point as shown in the diagram.



The angle between two vectors

Question: How to calculate angle θ between two given vectors?

To calculate angle between two vectors, we can use either dot or cross product depending on the data in a given problem.

Angle from Dot-Product:

Using dot product, $\vec{a}.\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}.\vec{b}}{\|\vec{a}\|\|\vec{b}\|}\right)$

Angle from Cross-Product: In vector form, $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta\,\hat{\mathbf{n}}$.

Taking norm both sides, we have $\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\||\sin\theta| \Rightarrow |\sin\theta| = \left(\frac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|\|\vec{b}\|}\right)$

These formula are useful to compute angle between any two non-zero vectors.

Important Properties of Norms: Suppose $\vec{a}$ and $\vec{b}$ are non-zero vectors.

a) $\|\vec{a}\|^2 = \vec{a}.\vec{a} = \vec{a}^2$ & $\|\vec{a}\| = \sqrt{\vec{a}.\vec{a}}$

b) $\|k\vec{a}\| = |k|\|\vec{a}\|$

c) $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 \Leftrightarrow \vec{a} \perp \vec{b}$

d) $\|\vec{a}+\vec{b}\| = \|\vec{a}-\vec{b}\| \Leftrightarrow \vec{a}.\vec{b} = 0$

e) $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\|\vec{a}\|\|\vec{b}\|\cos\theta$

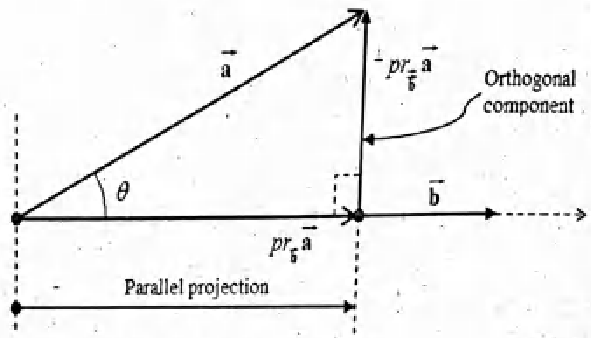
f) $\|\vec{a}-\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$

Projection of Vectors:

For the non-zero vectors $\vec{a}$ and $\vec{b}$, there are components one vector over the other vector. These components are generally known as projections.

Scalar projections: Consider the following diagram.

Once the angle θ between the vectors is known, then the scalar values of the projections are given as $pr_{\vec{b}}\vec{a} = \|\vec{a}\|\cos\theta$ and $^{\perp}pr_{\vec{b}}\vec{a} = \|\vec{a}\|\sin\theta$.



Projections of a onto b

Vector Projections:

Parallel projection (component):

The component of $\vec{a}$ along $\vec{b}$ or parallel to $\vec{b}$ is called parallel projection of $\vec{a}$ onto $\vec{b}$. It is given by $pr_{\vec{b}}\vec{a} = \left(\frac{\vec{a}.\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$.

Orthogonal component (projection):

A vector connecting the tip of $\vec{a}$ to some point perpendicular to $\vec{b}$ is known as orthogonal (perpendicular) component or projection. It is denoted with a perpendicular symbol as $^{\perp}pr_{\vec{b}}\vec{a}$. The mathematical formula for the orthogonal projection $^{\perp}pr_{\vec{b}}\vec{a}$ is obtained from triangle rule for vectors.

That is $pr_{\vec{b}}\vec{a} + {}^{\perp}pr_{\vec{b}}\vec{a} = \vec{a} \Rightarrow {}^{\perp}pr_{\vec{b}}\vec{a} = \vec{a} - pr_{\vec{b}}\vec{a} \Rightarrow {}^{\perp}pr_{\vec{b}}\vec{a} = \vec{a} - \left(\frac{\vec{a}.\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}$.

Examples:

1. Find the angle between the vectors $\vec{a} = i + k$ and $\vec{b} = j + k$.

Solution: Here, $\|a\| = \sqrt{2}$, $\|b\| = \sqrt{2}$, $a.b = 1 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

2. Find c for which the angle between $\vec{a} = i - 2cj + k$, $\vec{b} = i + j$ is $\pi/4$.

Solution: Here,

$$\vec{a}.\vec{b} = 1 - 2c, \|\vec{a}\| = \sqrt{4c^2 + 2}, \|\vec{b}\| = \sqrt{2} \Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{\|\vec{a}\|\|\vec{b}\|}$$

$$\Rightarrow \frac{1 - 2c}{\sqrt{2}\sqrt{4c^2 + 2}} = \frac{1}{\sqrt{2}} \Rightarrow 4c^2 + 2 = 4c^2 - 4c + 1 \Rightarrow -4c = 1 \Rightarrow c = -\frac{1}{4}$$

3. Let $\vec{a} = 2i - 4j + 10k$ and $\vec{b} = i + j + 2k$. Find the parallel and orthogonal projections of $\vec{a}$ onto $\vec{b}$, $\Pr_{\vec{b}}^{\vec{a}}$ and ${}^{\perp}\Pr_{\vec{b}}^{\vec{a}}$

Solution: Here, $\vec{a}.\vec{b} = 2 - 4 + 20 = 18$, $\|\vec{b}\| = \sqrt{6}$.

Thus, the parallel projection is $pr_{\vec{b}}\vec{a} = \left(\frac{\vec{a}.\vec{b}}{\|\vec{b}\|^2}\right)\vec{b} = (\frac{18}{6})\vec{b} = 3i + 3j + 6k$.

The orthogonal projection of $\vec{a}$ onto $\vec{b}$ is ${}^{\perp}pr_{\vec{b}}\vec{a} = \vec{a} - pr_{\vec{b}}\vec{a} = -i - 7j + 4k$.

4. Let $\vec{a} = i + 2j + k, \vec{b} = 3i - 2j - k, \vec{c} = i - j + k$. Then, find $pr_{2\vec{c}}(\vec{a} + \vec{b})$

Solution: Here, $\vec{a} + \vec{b} = 4i, 2\vec{c} = 2i - 2j + 2k$, $\|2\vec{c}\| = \sqrt{12}$.

Hence, $pr_{2\vec{c}}(\vec{a} + \vec{b}) = \left(\frac{(\vec{a}+\vec{b}).2\vec{c}}{\|2\vec{c}\|^2}\right)2\vec{c} = \frac{8}{12}(2 - 2j + 2k) = \frac{2}{3}(2 - 2j + 2k)$.

5. Let $\vec{a} = (1,2,3,4,5)$, $\vec{b} = (6,7,8,9,-10)$. Find the projection of $\vec{b}$ onto $\vec{a}$.

Solution: $pr_{\vec{a}}\vec{b} = \left(\frac{\vec{a}.\vec{b}}{\|\vec{a}\|^2}\right)\vec{a} = \left(\frac{30}{55}\right)(1,2,3,4,5) = \left(\frac{6}{11}\right)(1,2,3,4,5)$.

6. Finding Interior Angles of a triangle: For a triangle whose vertices are $A=(-3,5,6), B=(-2,7,9)$ and $C=(2,1,7)$, find all the three interior angles.

Solution: First, find the three vectors representing the sides of the triangle. Be careful about the orders when you determine the vectors.

Here, $\overrightarrow{AB}=B-A=(1,2,3), \overrightarrow{AC}=C-A=(5,-4,1), \overrightarrow{BC}=C-B=(4,-6,-2)$.

Then, $\|\overrightarrow{AB}\|=\sqrt{14}, \|\overrightarrow{AC}\|=\sqrt{42}, \|\overrightarrow{BC}\|=2\sqrt{14}$.

So, using the dot product, the interior angles are determined as follow:

$$\begin{cases} \cos\angle A=\frac{\overrightarrow{AB}.\overrightarrow{AC}}{\|\overrightarrow{AB}\|\|\overrightarrow{AC}\|}=\frac{5-8+3}{\sqrt{14}\sqrt{42}}=0 \Rightarrow \cos\angle A=0 \Rightarrow \angle A=90^0 \\ \cos\angle B=\frac{\overrightarrow{AB}.\overrightarrow{BC}}{\|\overrightarrow{AB}\|\|\overrightarrow{BC}\|}=\frac{4-12-6}{\sqrt{14}.2\sqrt{14}}=-\frac{14}{28}=-\frac{1}{2} \Rightarrow \cos\angle B=-\frac{1}{2} \Rightarrow \angle B=120^0 \\ \cos\angle C=\frac{\overrightarrow{AC}.\overrightarrow{BC}}{\|\overrightarrow{AC}\|\|\overrightarrow{BC}\|}=\frac{20+24-2}{\sqrt{42}.2\sqrt{14}}=\frac{42}{28\sqrt{3}}=\frac{\sqrt{3}}{2} \Rightarrow \cos\angle B=\frac{\sqrt{3}}{2} \Rightarrow \angle B=30^0 \end{cases}$$

But the sum $\angle A+\angle B+\angle C=240^0$ which is impossible.

Question: What is the **BIG** mistake committed in this solution?

Anyway, the correct interior angles are $\angle A=90^0$, $\angle B=60^0$ and $\angle C=30^0$.

7. If $\vec{a}$ and $\vec{b}$ are orthogonal unit vectors, show that $\|\vec{a}+\vec{b}\|=\|\vec{a}-\vec{b}\|=\sqrt{2}$.

Solution: Since $\vec{a}$ and $\vec{b}$ are orthogonal unit vectors, $\|\vec{a}\|=\|\vec{b}\|=1$ and $\vec{a}.\vec{b}=0$.

Hence, *i)* $\|\vec{a}+\vec{b}\|^2=(\vec{a}+\vec{b})(\vec{a}+\vec{b})=\|\vec{a}\|^2+\|\vec{b}\|^2+2\vec{a}.\vec{b}=2 \Rightarrow \|\vec{a}+\vec{b}\|=\sqrt{2}$

ii) $\|\vec{a}-\vec{b}\|^2=(\vec{a}-\vec{b})(\vec{a}-\vec{b})=\|\vec{a}\|^2+\|\vec{b}\|^2-2\vec{a}.\vec{b}=2 \Rightarrow \|\vec{a}-\vec{b}\|=\sqrt{2}$

8. Suppose $\vec{a}.\vec{b}=-12, \|\vec{b}\|=4$. Find the norm $\|pr_{\vec{b}}\vec{a}\|$.

Solution: $\|pr_{\vec{b}}\vec{a}\|=\left\|\left(\frac{\vec{a}.\vec{b}}{\|\vec{b}\|^2}\right)\vec{b}\right\|=\left|\frac{\vec{a}.\vec{b}}{\|\vec{b}\|^2}\right|\|\vec{b}\|=\frac{|\vec{a}.\vec{b}|}{\|\vec{b}\|}=\frac{|-12|}{4}=3$.

9. Given the vectors with $\|\vec{a}\| = 2, \|\vec{b}\| = 2\sqrt{7}, \vec{a}.\vec{b} = 2$. Then, find the angle between the vectors $\vec{a}$ and $\vec{a}+\vec{b}$.

Solution: Let the angle between $\vec{a}$ and $\vec{a}+\vec{b}$ be θ.

But, $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + 2\vec{a}.\vec{b} + \|\vec{b}\|^2 = 4+28+4 = 36 \Rightarrow \|\vec{a}+\vec{b}\| = 6$. Hence,

$$\cos\theta = \frac{\vec{a}.(\vec{a}+\vec{b})}{\|\vec{a}\|\|\vec{a}+\vec{b}\|} = \frac{\vec{a}.\vec{a}+\vec{a}.\vec{b}}{\|\vec{a}\|\|\vec{a}+\vec{b}\|} = \frac{\|\vec{a}\|^2 + \vec{a}.\vec{b}}{\|\vec{a}\|\|\vec{a}+\vec{b}\|} = \frac{4+2}{2(6)} = \frac{1}{2} \Rightarrow \theta = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$$

10. Given $\vec{b} = 6i+5j+4k$ and ${}^{\perp}pr_{\vec{a}}\vec{b} = 2i+j+6k$. If $\|\vec{a}\| = 3$, find $pr_{\vec{a}}\vec{b}$ and $\vec{a}$.

Solution: First observe the geometric relation among $\vec{b}$, $pr_{\vec{a}}\vec{b}$ and ${}^{\perp}pr_{\vec{a}}\vec{b}$.

From vector addition, we have

i) $pr_{\vec{a}}\vec{b} + {}^{\perp}pr_{\vec{a}}\vec{b} = \vec{b} \Rightarrow pr_{\vec{a}}\vec{b} = \vec{b} - {}^{\perp}pr_{\vec{a}}\vec{b} = 4i+4j-2k$

ii) $\vec{a} \,//\, pr_{\vec{a}}\vec{b} \Rightarrow \vec{a} = t.pr_{\vec{a}}\vec{b} \Rightarrow \|\vec{a}\| = \|t pr_{\vec{a}}\vec{b}\| = |t|\|pr_{\vec{a}}\vec{b}\|$

$$\Rightarrow |t| = \tfrac{3}{6} = \tfrac{1}{2} \Rightarrow t = \pm\tfrac{1}{2} \Rightarrow \vec{a} = \pm\tfrac{1}{2}\, pr_{\vec{a}}\vec{b} = \pm(2i+2j-k)$$

11. Suppose $\vec{a}$ and $\vec{b}$ are nonzero vectors and the angle between them is θ with $0 \le \theta \le \pi$. If $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\|$, find angle θ.

Solution: $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\| \Rightarrow \|\vec{a}\|\|\vec{b}\| = \|\vec{a}\|\|\vec{b}\|\sin\theta \Rightarrow \sin\theta = 1 \Rightarrow \theta = \sin^{-1}(1) = \frac{\pi}{2}$.

12. If $\vec{a}\times\vec{b} = i+2j+2k, \|\vec{a}\| = \|\vec{b}\| = \sqrt{5}$, find the cosine value of the angle between the vectors $\vec{a}$ and $\vec{b}$.

Solution: Here, $\vec{a}\times\vec{b} = i+2j+2k \Rightarrow \|\vec{a}\times\vec{b}\| = \|i+2j+2k\| = 3$. Then,

$$\|\vec{a}\times\vec{b}\| = \|\vec{a}\|\|\vec{b}\||\sin\theta| \Rightarrow \sqrt{5}.\sqrt{5}|\sin\theta| = 3 \Rightarrow 5|\sin\theta| = 3 \Rightarrow |\sin\theta| = \tfrac{3}{5}$$

Hence, using trigonometric identity, we have

$$\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos^2\theta = 1-\sin^2\theta = 1-\tfrac{9}{25} = \tfrac{16}{25} \Rightarrow \cos\theta = \pm\tfrac{4}{5}.$$

13. Given $\|\vec{a}\| = 3\sqrt{6}, \|\vec{b}\| = 12$ and $pr_{\vec{b}}\vec{a} = 2i + 4j - 4k$. Then, find $\|pr_{\vec{a}}\vec{b}\|$ and give the cosine value of angle θ between $\vec{a}$ and $\vec{b}$.

Solution: Given the projection vector $pr_{\vec{b}}\vec{a} = 2i + 4j - 4k$.

Using the projection formula, $pr_{\vec{b}}\vec{a} = (\frac{\vec{a}.\vec{b}}{\|\vec{b}\|^2})\vec{b} = 2i + 4j - 4k$.

Then, take norm on both sides to get the value of $\vec{a}.\vec{b}$.

$$\left\|(\frac{\vec{a}.\vec{b}}{\|\vec{b}\|^2})\vec{b}\right\| = \|2i + 4j - 4k\| \Rightarrow \frac{|\vec{a}.\vec{b}|}{\|\vec{b}\|^2}\|\vec{b}\| = \|2i + 4j - 4k\| \Rightarrow \frac{|\vec{a}.\vec{b}|}{\|\vec{b}\|} = 6$$

$$\Rightarrow |\vec{a}.\vec{b}| = 6\|\vec{b}\| \Rightarrow |\vec{a}.\vec{b}| = 6(12) \Rightarrow |\vec{a}.\vec{b}| = 72$$

Here, using absolute value property, $|\vec{a}.\vec{b}| = 72 \Rightarrow \vec{a}.\vec{b} = \pm 72$.

Thus, $\|pr_{\vec{a}}\vec{b}\| = \left\|(\frac{\vec{a}.\vec{b}}{\|\vec{a}\|^2})\vec{a}\right\| = \frac{|\vec{a}.\vec{b}|}{\|\vec{a}\|^2}\|\vec{a}\| = \frac{|\vec{a}.\vec{b}|}{\|\vec{a}\|} = \frac{72}{3\sqrt{6}} = \frac{24}{\sqrt{6}} = 4\sqrt{6}$.

Besides, $\cos\theta = \frac{\vec{a}.\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \pm\frac{72}{(3\sqrt{6})(12)} = \pm\frac{2}{\sqrt{6}} = \pm\frac{\sqrt{6}}{3}$ since $|\vec{a}.\vec{b}| = 72 \Rightarrow \vec{a}.\vec{b} = \pm 72$.

Notice that we use the minus sign if the angle between the vectors is obtuse.

14. Given $\|\vec{a} + \vec{b}\| = 2$ where $\vec{a}$ and $\vec{b}$ are unit vectors. Find $\|\vec{a} + 4\vec{b}\|$.

Solution: Here, $\|\vec{a} + \vec{b}\| = 2 \Rightarrow \|\vec{a} + \vec{b}\|^2 = 4 \Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = 4$

$$\Rightarrow \vec{a}.\vec{a} + 2\vec{a}.\vec{b} + \vec{b}.\vec{b} = 4 \Rightarrow \|\vec{a}\|^2 + 2\vec{a}.\vec{b} + \|\vec{b}\|^2 = 4$$

$$\Rightarrow 1 + 2\vec{a}.\vec{b} + 1 = 4 \Rightarrow \vec{a}.\vec{b} = 1$$

Now, $\|\vec{a} + 4\vec{b}\|^2 = (\vec{a} + 4\vec{b}).(\vec{a} + 4\vec{b}) = \vec{a}.\vec{a} + 8\vec{a}.\vec{b} + 16\vec{b}.\vec{b}$

$$= \|\vec{a}\|^2 + 8\vec{a}.\vec{b} + 16\|\vec{b}\|^2 = 1 + 8 + 16 = 25$$

$$\Rightarrow \|\vec{a} + 4\vec{b}\|^2 = 25 \Rightarrow \|\vec{a} + 4\vec{b}\| = 5$$

15. Given $\|\vec{a}\| = 3\sqrt{6}, \|\vec{b}\| = 12, pr_{\vec{b}}\vec{a} = 2i + 4j - 4k$ and the orthogonal projection of $\vec{a}$ onto $\vec{b}$ is $^{\perp}pr_{\vec{a}}\vec{b} = -4i + 4j - 4k$. Find $\vec{a}, \vec{b}$ and $pr_{\vec{a}}\vec{b}$.

Solution: Given $pr_{\vec{b}}\vec{a} = 2i + 4j - 4k$ and $^{\perp}pr_{\vec{a}}\vec{b} = -4i + 4j - 4k$.

Parallel Relation:

But from the concept of projection, always $pr_{\vec{b}}\vec{a}$ is parallel to vector $\vec{b}$.

Similarly, $pr_{\vec{a}}\vec{b}$ is parallel to vector $\vec{a}$.

So, there exists some scalar t such that $\vec{b} = t.pr_{\vec{b}}\vec{a}$.

But $\vec{b} = t.pr_{\vec{b}}\vec{a} \Rightarrow \|\vec{b}\| = \|t.pr_{\vec{b}}\vec{a}\| \Rightarrow \|\vec{b}\| = |t|\|pr_{\vec{b}}\vec{a}\| \Rightarrow |t| = \frac{\|\vec{b}\|}{\|pr_{\vec{b}}\vec{a}\|} = \frac{12}{6} = 2 \Rightarrow t = \pm 2$.

Therefore, $\vec{b} = t.pr_{\vec{b}}\vec{a} = \pm 2(2i + 4j - 4k) = \pm(4i + 8j - 8k)$

Triangle Relation:

Always the parallel and orthogonal projections of a vector form a right angle triangle where the hypotenuse is along the vector projected.

That is $pr_{\vec{b}}\vec{a} + {}^{\perp}pr_{\vec{b}}\vec{a} = \vec{a}$. Similarly, $pr_{\vec{a}}\vec{b} + {}^{\perp}pr_{\vec{a}}\vec{b} = \vec{b}$.

So, use this relation to get the vectors $\vec{a}$ and $pr_{\vec{a}}\vec{b}$ as follow:

Case-1: Assume the angle between the vectors is acute. That means $\vec{a}.\vec{b} > 0$.

Here, $pr_{\vec{a}}\vec{b} + {}^{\perp}pr_{\vec{a}}\vec{b} = \vec{b} \Rightarrow pr_{\vec{a}}\vec{b} = \vec{b} - {}^{\perp}pr_{\vec{a}}\vec{b} \Rightarrow pr_{\vec{a}}\vec{b} = 8i + 4j - 4k$.

Again, use the parallel relation stated above to get the scalar:

That is $\vec{a} = t.pr_{\vec{a}}\vec{b} \Rightarrow \|\vec{a}\| = \|t.pr_{\vec{a}}\vec{b}\| \Rightarrow |t| = \frac{\|\vec{a}\|}{\|pr_{\vec{a}}\vec{b}\|} = \frac{3\sqrt{6}}{\sqrt{96}} = \frac{3\sqrt{6}}{4\sqrt{6}} = \frac{3}{4}$.

Therefore, for the acute case, $\vec{a} = t.pr_{\vec{a}}\vec{b} = \frac{3}{4}(8i + 4j - 4k) = 6i + 3j - 3k$.

Case-2: Assume the angle between the vectors is obtuse. That is $\vec{a}.\vec{b} < 0$.

Using similar argument, we get $pr_{\vec{a}}\vec{b} = -8i - 4j + 4k$ and $\vec{a} = -6i - 3j + 3k$.

16. If $\vec{u} = i + j + k$, find vector $\vec{x}$ with the property $\vec{u} \times \vec{x} = i - j$ and $\|\vec{x}\| = \sqrt{2}$.

Solution: If $\vec{x} = ai + bj + ck$, find $\vec{u} \times \vec{x}$ and equate with $\vec{u} \times \vec{x} = i - j$.

That is $\vec{u} \times \vec{x} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ a & b & c \end{vmatrix} = (c-b)\mathbf{i} + (a-c)\mathbf{j} + (b-a)\mathbf{k} = \mathbf{i} - \mathbf{j}$

$$\Rightarrow c - b = 1, a - c = -1, b - a = 0$$

$$\Rightarrow c = b + 1, c = a + 1, b = a \Rightarrow b = a, c = a + 1$$

Then, put the relation $b = a, c = a + 1$ in the given value $\|\vec{x}\| = \sqrt{2}$.

That is $\|\vec{x}\| = \sqrt{2} \Rightarrow \sqrt{a^2 + b^2 + c^2} = \sqrt{2} \Rightarrow \sqrt{a^2 + a^2 + (a+1)^2} = \sqrt{2}$

$$\Rightarrow \sqrt{3a^2 + 2a + 1} = \sqrt{2} \Rightarrow 3a^2 + 2a + 1 = 2$$

$$\Rightarrow 3a^2 + 2a - 1 = 0 \Rightarrow a = \tfrac{1}{3}, a = -1$$

Therefore, the vector is $\vec{x} = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}$ or $\vec{x} = -\mathbf{i} - \mathbf{j}$.

17. Given $\vec{a}$, $\vec{b}$ and $\vec{c}$ with $\vec{a} + \vec{b} + \vec{c} = 0$, $\|\vec{a}\| = 2$, $\|\vec{b}\| = 4$ and $\|\vec{c}\| = 6$. Then,

a) Determine the angle between $\vec{a}$ and $\vec{b}$

b) Find the value of $\vec{a}.\vec{b} + \vec{a}.\vec{b} + \vec{b}.\vec{c}$

Solution: If two vectors are equal, then their norms are equal. So,

a) $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \vec{a} + \vec{b} = -\vec{c} \Rightarrow \|\vec{a} + \vec{b}\|^2 = \|-\vec{c}\|^2 \Rightarrow (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = \vec{c}.\vec{c}$

$$\Rightarrow \|\vec{a}\|^2 + \|\vec{b}\|^2 + 2\vec{a}.\vec{b} = \|\vec{c}\|^2 \Rightarrow 4 + 16 + 2\|\vec{a}\|\|\vec{b}\|\cos\theta = 36$$

$$\Rightarrow \cos\theta = 1 \Rightarrow \theta = \cos^{-1}(1) = 0$$

b) $\vec{a} + \vec{b} + \vec{c} = 0 \Rightarrow \|\vec{a} + \vec{b} + \vec{c}\|^2 = \|0\| \Rightarrow (\vec{a} + \vec{b} + \vec{c}).(\vec{a} + \vec{b} + \vec{c}) = 0$

$$\Rightarrow \vec{a}.\vec{b} + \vec{a}.\vec{c} + \vec{b}.\vec{c} = -\frac{1}{2}(\|\vec{a}\|^2 + \|\vec{b}\|^2 + \|\vec{c}\|^2) = -28$$

18. Suppose $\|\vec{a}\| = 3$, $\|\vec{b}\| = 4, \vec{a}.\vec{b} = 4$. Then, find $\|\vec{a}\times\vec{b}\|$.

Solution: Let θ be the angle between the vectors. Then,

$\vec{a}.\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta \Rightarrow 12\cos\theta = 4 \Rightarrow \cos\theta = \frac{4}{12} = \frac{1}{3}$.

So, $\cos^2\theta + \sin^2\theta = 1 \Rightarrow \sin^2\theta = 1-\cos^2\theta = 1-\frac{1}{9} = \frac{8}{9} \Rightarrow \sin\theta = \pm\frac{2\sqrt{2}}{3}$

Hence, $\|\vec{a}\times\vec{b}\| = \|\vec{a}\|.\|\vec{b}\|.|\sin\theta| = 3(4)(\frac{2\sqrt{2}}{3}) = 8\sqrt{2}$.

19. Let $\vec{a} = 2i+3j+5k$, $\vec{b} = i+2j-k, \vec{c} = i-j+2k$. If $\vec{a}+t\vec{b}$ is perpendicular to $\vec{c}$, find the value of the scalar t.

Solution: Here,

$(\vec{a}+t\vec{b}) \perp \vec{c} \Rightarrow (\vec{a}+t\vec{b}).\vec{c} = 0 \Rightarrow 2+t-3-2t+10-2t = 0 \Rightarrow -3t = -9 \Rightarrow t = 3$.

20. If $\vec{a}+\vec{b}+\vec{c} = 0$ for three vectors $\vec{a},\vec{b}$ and $\vec{c}$, show that $\vec{a}\times\vec{b} = \vec{b}\times\vec{c} = \vec{c}\times\vec{a}$

Solution: $\vec{a}+\vec{b}+\vec{c} = 0 \Rightarrow \vec{a}+\vec{b} = -\vec{c}$. Then, take cross product both sides.

Using $\vec{a}$, $\vec{a}\times(\vec{a}+\vec{b}) = \vec{a}\times(-\vec{c}) \Rightarrow \vec{a}\times\vec{a}+\vec{a}\times\vec{b} = -\vec{a}\times\vec{c} \Rightarrow \vec{a}\times\vec{b} = \vec{c}\times\vec{a}$.

Using $\vec{c}$, $\vec{c}\times(\vec{a}+\vec{b}) = \vec{c}\times(-\vec{c}) \Rightarrow \vec{c}\times\vec{a}+\vec{c}\times\vec{b} = -\vec{c}\times\vec{c} \Rightarrow \vec{c}\times\vec{a} = \vec{b}\times\vec{c}$.

Therefore. From the first and the third, $\vec{a}\times\vec{b} = \vec{c}\times\vec{a} = \vec{b}\times\vec{c}$.

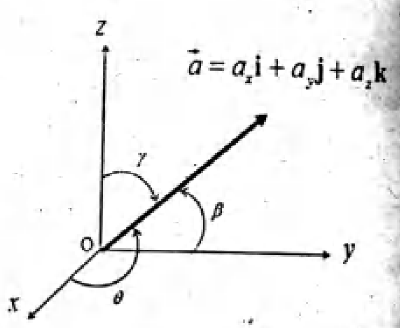
2.6 Direction Angles and Direction Cosines

Direction Angles and Direction Cosines:

The measures of the angles α,β,γ in $[0,\pi]$ formed by a non-zero vector $\vec{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and the positive coordinate axes are called *direction angles.* The coordinates formed by the cosines $(\cos\alpha, \cos\beta, \cos\gamma)$ of the direction angles are called *direction cosines.* The components a_x, a_y, a_z of $\vec{a}$ and its direction angles α,β,γ are related as follow:

Mathematical Relation among the Components and Direction cosines:

$$\begin{cases} a_x = \|a\|\cos\alpha \Rightarrow \cos\alpha = \frac{a_x}{\|a\|} \\ a_y = \|a\|\cos\beta \Rightarrow \cos\beta = \frac{a_y}{\|a\|} \\ a_z = \|a\|\cos\gamma \Rightarrow \cos\gamma = \frac{a_z}{\|a\|} \end{cases}$$



Direction Angles of a vector

Important Identity:

For direction angles α,β,γ of $\vec{a}$, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. (How?)

Justification: First, recall that $\vec{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \Rightarrow \|a\|^2 = a_x^2 + a_y^2 + a_z^2$

Now, square the cosine value expressions of the direction angles.

$$\text{That is} \begin{cases} a_x = \|a\|\cos\alpha \Rightarrow \cos\alpha = \frac{a_x}{\|a\|} \Rightarrow \cos^2\alpha = \frac{a_x^2}{\|a\|^2} \\ a_y = \|a\|\cos\beta \Rightarrow \cos\beta = \frac{a_y}{\|a\|} \Rightarrow \cos^2\beta = \frac{a_y^2}{\|a\|^2} \\ a_z = \|a\|\cos\gamma \Rightarrow \cos\gamma = \frac{a_z}{\|a\|} \Rightarrow \cos^2\gamma = \frac{a_z^2}{\|a\|^2} \end{cases}$$

Therefore, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{a_x^2 + a_y^2 + a_z^2}{\|a\|^2} = \frac{\|a\|^2}{\|a\|^2} = 1$.

Usefulness of the Identity: $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

We can use this identity if the magnitude of a vector and its direction angles are known, then the vector can be obtained by determining the components from the identity.

That is $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = \|a\|\cos\alpha.\mathbf{i} + \|a\|\cos\beta.\mathbf{j} + \|a\|\cos\gamma.\mathbf{k}$.

Examples:

1. Find the direction cosines and angles of a vector $\vec{a} = 10\mathbf{i} - 10\mathbf{k}$

Solution: Since $\vec{a}$ is given in vector form, the direction cosines and direction angles are obtained from the above identity as follow:

$$\begin{cases} \cos\alpha = \frac{a_x}{\|\vec{a}\|} = \frac{10}{10\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^0 = \frac{\pi}{4} \\ \cos\beta = \frac{a_y}{\|\vec{a}\|} = 0 \Rightarrow \beta = \cos^{-1}(0) = 90^0 = \frac{\pi}{2} \\ \cos\gamma = \frac{a_z}{\|\vec{a}\|} = -\frac{10}{10\sqrt{2}} = -\frac{1}{\sqrt{2}} \Rightarrow \gamma = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 135^0 = \frac{3\pi}{4} \end{cases}$$

2. Let α, β, γ be the direction angles of a vector $\vec{a} = a_x i + a_y j + a_z k$.

Then, find the numerical value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$.

Solution: Let's use $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. From trigonometry, we know $\sin^2\theta + \cos^2\theta = 1$ for any angle θ.

So, using this identity, we have

$$\begin{cases} \sin^2\alpha + \cos^2\alpha = 1 \Rightarrow \sin^2\alpha = 1 - \cos^2\alpha \\ \sin^2\beta + \cos^2\beta = 1 \Rightarrow \sin^2\beta = 1 - \cos^2\beta \\ \sin^2\gamma + \cos^2\gamma = 1 \Rightarrow \sin^2\gamma = 1 - \cos^2\gamma \end{cases}$$

Therefore, add the values in terms of cosines as follow:

$$\begin{aligned} \sin^2\alpha + \sin^2\beta + \sin^2\gamma &= 1 - \cos^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma \\ &= 3 - \underbrace{(\cos^2\alpha + \cos^2\beta + \cos^2\gamma)}_{=1} = 3 - 1 = 2 \end{aligned}$$

3. Suppose $\vec{v}$ is a vector in space where $v_x = 3$ and $v_y = 3$. If α, β and γ are its direction angles with $\gamma = 45^0$, then find vector $\vec{v}$ and give its direction cosines.

Solution: By definition, $\cos\alpha = \frac{v_x}{|v|}, \cos\beta = \frac{v_y}{|v|}, \cos\gamma = \frac{v_z}{|v|}$. Then, using the given values, $\cos\alpha = \frac{3}{|v|}, \cos\beta = \frac{3}{|v|}, \cos\gamma = \cos 45^0 = \frac{\sqrt{2}}{2}$.

Now, use these values in the identify as follow:

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow (\frac{3}{|v|})^2 + (\frac{3}{|v|})^2 + \cos^2(45^0) = 1$

$$\Rightarrow \frac{9}{|v|^2} + \frac{9}{|v|^2} + \frac{1}{2} = 1 \Rightarrow \frac{18}{|v|^2} = \frac{1}{2} \Rightarrow \|v\| = 6$$

Besides, using $\gamma = 45^0$ and $\|v\| = 6$, we get $v_z = \|v\|\cos\gamma = 6\cos(45^0) = 3\sqrt{2}$.

Hence, the vector is $\vec{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k} = 3\mathbf{i} + 3\mathbf{j} + 3\sqrt{2}\mathbf{k}$.

Its direction cosines: $\cos\alpha = \frac{v_x}{|v|} = \frac{1}{2}, \cos\beta = \frac{v_y}{|v|} = \frac{1}{2}, \cos\gamma = \frac{v_z}{|v|} = \frac{\sqrt{2}}{2}$

4. Find the direction cosines of a vector that makes equal angles with the coordinate axes.

Solution: Suppose α, β and γ are the direction angles such that $\alpha = \beta = \gamma$.

Then, $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}$.

5. Suppose a certain vector $\vec{a}$ is found in the first octant such that $\|\vec{a}\| = 35$ and the direction cosines with respect to x and y axes are $\frac{3}{7}$ and $\frac{2}{7}$ respectively. Find the direction cosine with respect to z axis and determine vector $\vec{a}$.

Solution: Use the cosine direction identify as follow:

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\Rightarrow (\tfrac{3}{7})^2 + (\tfrac{2}{7})^2 + \cos^2\gamma = 1$$

$$\Rightarrow \tfrac{9}{49} + \tfrac{4}{49} + \cos^2\gamma = 1 \Rightarrow \cos^2\gamma = \tfrac{36}{49} \Rightarrow \cos\gamma = \pm\tfrac{6}{7}$$

Since the vector is in the first octant only $\cos\gamma = \frac{6}{7}$ is valid.

Moreover, $a_x = \|\vec{a}\|\cos\alpha = 35(\frac{3}{7}) = 15, a_y = 35(\frac{2}{7}) = 10, a_z = 35(\frac{6}{7}) = 30$.

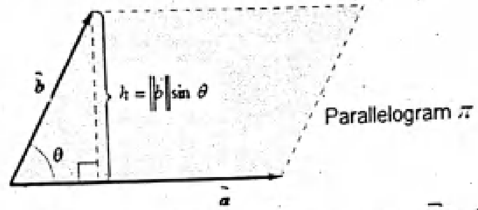
Therefore, the vector becomes $\vec{a} = a_x i + a_y j + a_z k = 15i + 10j + 30k$.

2.7 Applications of Vectors

2.7.1 Computations of Area and Volume

Now, let's see how to apply vectors in the computation of area and volume.

I) Computation of Area: Let $PQRT$ be a parallelogram whose adjacent sides are given by the vectors $\vec{a}$ and $\vec{b}$ as shown in the figure below.



Area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

Now let's calculate the area of the parallelogram π denoted by $A(\pi)$.

Here, Area $(PQRT) = B.h$ where B is the base of the parallelogram $PQRT$ given by $B=\|\vec{a}\|$ and h is the altitude.

By trigonometric relations, we have $h=\|\vec{b}\|\sin\theta$. But $\sin\theta=\dfrac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|.\|\vec{b}\|}$.

Thus, Area $(PQRT) = B.h = \|\vec{a}\|.\|\vec{b}\|\sin\theta = \|\vec{a}\|\|\vec{b}\|.\dfrac{\|\vec{a}\times\vec{b}\|}{\|\vec{a}\|.\|\vec{b}\|} = \|\vec{a}\times\vec{b}\|$.

Therefore, the area of a parallelogram whose adjacent sides are the vectors $\vec{a}$ and $\vec{b}$ is computed by using Area $=\|\vec{a}\times\vec{b}\|$.

Note that if the figure was a triangle whose adjacent sides are the vectors $\vec{a}$ and $\vec{b}$, the area is given by Area $=\frac{1}{2}\|\vec{a}\times\vec{b}\|$.

Examples: Find the area of

a) The parallelogram whose adjacent sides are given by the vectors

$\vec{a} = i + 2j + 3k$ and $\vec{b} = 4i + 5j + 6k$

b) The triangle whose adjacent sides are given by the vectors

$\vec{a} = i - 3j + 2k$ and $\vec{b} = 2i + 2j - 2k$.

Solution:

a) Here, $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = -3i + 6j - 3k$. Thus, $A(\pi) = \|\vec{a} \times \vec{b}\| = 3\sqrt{6}$.

b) Here, $\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -3 & 2 \\ 2 & 2 & -2 \end{vmatrix} = 2i + 6j + 8k$. So, $A = \frac{1}{2}\|\vec{a} \times \vec{b}\| = \sqrt{26}$.

Remark: Sometimes the sides of the parallelogram may not be given in terms of vectors rather the vertices of the parallelogram may be given. In such situations, first identify three vertices (one as common initial point) which give vectors along the adjacent sides. The problem here is how to identify the appropriate vertices that give vectors along adjacent sides of the parallelogram. This can be done in different ways. By finding the six vectors or by drawing the parallelogram or by computing three areas. Now let's see how we can identify the appropriate vectors from the six directed vectors formed by the vertices.
Suppose P, Q, R, S are vertices of the parallelogram π. Let
$\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{PS}, \overrightarrow{RQ}, \overrightarrow{RS}, \overrightarrow{SQ}$ be the directed vectors formed by the vertices.
From these six vectors, we can identify two pairs of equal (parallel, in case if they differ in sign) vectors, say such pair of equal (parallel) vectors are $\overrightarrow{PQ}, \overrightarrow{RS}$ and $\overrightarrow{PS}, \overrightarrow{RQ}$. (Can you find another pair which are parallel? No!). Now, take one vector from each pair, $\overrightarrow{PQ}, \overrightarrow{PS}$ or $\overrightarrow{PQ}, \overrightarrow{RQ}$ or $\overrightarrow{RS}, \overrightarrow{PS}$ or $\overrightarrow{RS}, \overrightarrow{RQ}$.

Examples: Find the area of

a) A parallelogram whose vertices are

$P(1,2,0)$, $Q(3,4,0)$, $R(4,1,0)$ and $S(2,-1,0)$.

b) A parallelogram $PQRT$ whose vertices are

$P=(2,1,3), Q=(1,4,5), R=(2,5,3), T=(3,2,1)$

c) A triangle whose vertices are $P(3,-2,0)$, $Q(2,2,2)$ and $R(-1,0,3)$.

Solution:

a) First let's find the six directed vectors formed by the four vertices.

$\overrightarrow{PQ}=2i+2j, \overrightarrow{PR}=3i-j, \overrightarrow{PS}=i-3j, \overrightarrow{RQ}=i-3j, \overrightarrow{RS}=2i+2j, \overrightarrow{SQ}=i+4j$ From these six vectors, the pair of equal (parallel) vectors are $\overrightarrow{PQ}=2i+2j$ with $\overrightarrow{RS}=2i+2j$ and $\overrightarrow{PR}=3i-j$ with $\overrightarrow{PS}=i-3j$. Hence, take $\overrightarrow{PQ}=2i+2j$ from the first and $\overrightarrow{PS}=i-3j$ from the second.

Here, $\overrightarrow{PQ}\times\overrightarrow{PS}=\begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 1 & -3 & 0 \end{vmatrix}=-8k$. Thus, $A(\pi)=\left\|\overrightarrow{PQ}\times\overrightarrow{PS}\right\|=8$.

b) Here, we are given that the parallelogram is labeled as $PQRT$. This means the adjacent vectors we need to compute the area are simply $\overrightarrow{PQ}$ and $\overrightarrow{PT}$.

Then, $\overrightarrow{PQ}=-i+3j+2k, \overrightarrow{PT}=i+j-2k$ and $\overrightarrow{PQ}\times\overrightarrow{PT}=\begin{vmatrix} i & j & k \\ -1 & 3 & 2 \\ 1 & 1 & -2 \end{vmatrix}=-8i-4k$

Therefore, $A=\left\|\overrightarrow{PQ}\times\overrightarrow{PT}\right\|=\|-8i-4k\|=\sqrt{64+16}=\sqrt{80}=4\sqrt{5}$

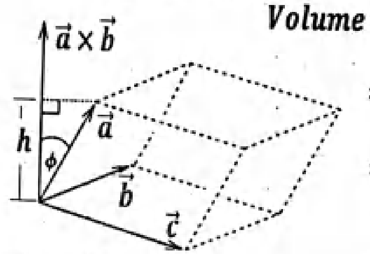
c) Since the figure is a triangle, simply find two vectors with common initial point using the three vertices. For instance by taking vertex R as common, we have $\overrightarrow{RP}=4i-2j-3k$, $\overrightarrow{RQ}=3i+2j-k$ such that

$\overrightarrow{RP}\times\overrightarrow{RQ}=\begin{vmatrix} i & j & k \\ 4 & -2 & -3 \\ 3 & 2 & -1 \end{vmatrix}=8i-5j+14k$. So, $A=1/2\left\|\overrightarrow{RP}\times\overrightarrow{RQ}\right\|=\sqrt{285}/2$

II) Computation of Volume:

Suppose $\vec{a}, \vec{b}$ and $\vec{c}$ are non-zero vectors with co-initial in space and consider a parallelepiped generated by these three vectors as shown in the diagram below.

Volume of Parallelpiped



$$Volume = B.h$$

$$= \|\vec{b}\times\vec{c}\|\|\vec{a}\|\cos\phi$$

$$= |\vec{a}.(\vec{b}\times\vec{c})|$$

$$= \left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right\|$$

$$h = \|\vec{a}\|\cos\phi$$

$$B = \|\vec{b}\times\vec{c}\|$$

Then the volume V of the parallelepiped is given by $V = Base\ Area \times height$.

But from the above discussion, the base area is given by $A = \|\vec{b}\times\vec{c}\|$.

Besides, from the right triangle $h = \|\vec{a}\|\cos\phi$.

Thus, $V = Base\ Area \times height = \|\vec{b}\times\vec{c}\|\|\vec{a}\|\cos\phi = |\vec{a}.(\vec{b}\times\vec{c})|$.

Remark: The quantities $\vec{a}.(\vec{b}\times\vec{c}), \vec{b}.(\vec{a}\times\vec{c})$ and $\vec{c}.(\vec{a}\times\vec{b})$ are called scalar triple product and all the three quantities are equal to each other in magnitude. Therefore, the volume of a parallelepiped determined by three non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ is given by the absolute value of their scalar triple product.

That is $V = |\vec{a}.(\vec{b}\times\vec{c})| = |\vec{b}.(\vec{a}\times\vec{c})| = |\vec{c}.(\vec{a}\times\vec{b})|$.

Suppose $\vec{a} = a_1 i + a_2 j + a_3 k, \vec{b} = b_1 i + b_2 j + b_3 k, \vec{c} = c_1 i + c_2 j + c_3 k$. Then, their scalar triple product is calculated using determinant as $\vec{a}.(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

From this formula, it is simple to verify that $|\vec{a}.(\vec{b}\times\vec{c})| = |\vec{b}.(\vec{a}\times\vec{c})| = |\vec{c}.(\vec{a}\times\vec{b})|$.

Examples:

1. Find the volume of the parallelepiped generated by $\vec{a}=2i+j+3k, \vec{b}=-i+3j+2k$ and $\vec{c}=i+j-2k$.

Solution: Using the determinant, the volume is given by

$$V=\left|\vec{a}.(\vec{b}\times\vec{c})\right|=\left\|\begin{matrix}2 & 1 & 3\\ -1 & 3 & 2\\ 1 & 1 & -2\end{matrix}\right\|=\left|-16-12\right|=\left|-28\right|=28 \text{ units.}$$

2. If the volume of a parallelepiped generated by the vectors $\vec{a}=i+2j+3k, \vec{b}=4i+5j+6k$ and $\vec{c}=7i+8j+xk$ is 12, find x.

Solution: Using the determinant, the volume is given by

$$V=\left|\vec{a}.(\vec{b}\times\vec{c})\right|=\left\|\begin{matrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & x\end{matrix}\right\|=\left|5x-48-8x+84-9\right|=\left|27-3x\right|$$

Then, using the given volume, $\left|27-3x\right|=12 \Rightarrow 27-3x=\pm 12 \Rightarrow x=5, x=13$.

3. Find the volume of a the tetrahedron generated by three adjacent vectors $\vec{a}=6i+4j-8k, \vec{b}=2i-j+k$ and $\vec{c}=3i+j+k$.

Solution: Volume of tetrahedron: The volume of a tetrahedron generated by three adjacent vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is $V=\frac{1}{6}\left|\vec{a}.(\vec{b}\times\vec{c})\right|$.

Therefore, $V=\frac{1}{6}\left|\vec{a}.(\vec{b}\times\vec{c})\right|=\frac{1}{6}\left\|\begin{matrix}6 & 4 & -8\\ 2 & -1 & 1\\ 3 & 1 & 1\end{matrix}\right\|=\frac{1}{6}\left|-48\right|=8$

4. Find the volume of the box formed by the following three adjacent vectors $\vec{u}=2i-6j+2k$, $\vec{v}=4j-2k$ and $\vec{w}=2i+2j-4k$.

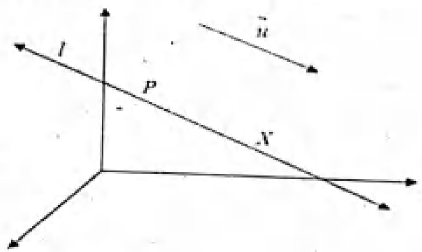
Solution : $V=\left|\vec{u}.(\vec{v}\times\vec{w})\right|=\left\|\begin{matrix}2 & -6 & 2\\ 0 & 4 & -2\\ 2 & 2 & -4\end{matrix}\right\|=\left|-24+24-16\right|=\left|-16\right|=16$

2.7.2 Equations of Lines and Planes in Space

From Euclidean geometry, we know that any two distinct points determine a unique line whose equation can be determined using the concepts in coordinate geometry. But, in space lines and planes are defined using the concept of vectors.

Lines in Space

Let l be a line in space through $P(x_0, y_0, z_0)$ parallel to the non-zero vector $\vec{u} = ai + bj + ck$ and suppose $X(x, y, z)$ is arbitrary point on this line (Refer the figure below). Since both P and X are on l the directed vector $\overrightarrow{PX}$ is parallel to l. Besides, the line is supposed to be parallel to vector $\vec{u}$. Hence, by transitivity of parallelism, the vectors $\overrightarrow{PX}$ and $\vec{u}$ are parallel. Therefore, $\overrightarrow{PX} = t\vec{u}$ for some scalar t. But $\overrightarrow{PX} = t\vec{u} \Rightarrow X - P = t\vec{u} \Rightarrow X = P + t\vec{u}$.



Using the above explanation, let's put the formal definition of a line in space.

Definition: The equation of a line l in space passing through the given point $P(x_0, y_0, z_0)$ and parallel to a non-zero vector $\vec{u} = ai + bj + ck$ is given by $l : X = P + t\vec{u}$, where t is a parameter. This is known as vector equation of a line and in this equation the vector $\vec{u}$ is called direction vector. Alternatively, if two points A and B of a line are given (instead of the direction vector), then the equation of l becomes $l : X = A + t\overrightarrow{AB}$, $t \in R$. Here, the vector $\overrightarrow{AB}$ is used as the direction *vector* of the line.

Parametric and Symmetric Equations of a line: Now, from the vector equation by letting $X = (x, y, z)$, $P = (x_0, y_0, z_0)$, $\vec{u} = (a, b, c)$, we get that the vector equation to be $l : (x, y, z) = (x_0, y_0, z_0) + t(a, b, c)$.

Equating corresponding components from this equation, we have

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$$. (This is called *Parametric Equation*).

Again, from the parametric equation (whenever a, b, c are all non zero), solving for the parameter t gives us,

$$\begin{cases} x = x_0 + ta \Rightarrow t = \dfrac{x - x_0}{a} \\ y = y_0 + tb \Rightarrow t = \dfrac{y - y_0}{b} \\ z = z_0 + tc \Rightarrow t = \dfrac{z - z_0}{c} \end{cases}$$

Equating these values of t gives $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$ (This is called *Symmetric Equation*).

The case when one of a, b, c is zero gives the following equations:

If $a = 0, x = x_0, \dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$, if $b = 0, y = y_0, \dfrac{x - x_0}{a} = \dfrac{z - z_0}{c}$ and

if $c = 0, z = z_0, \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$

Examples:

1. Give the vector, parametric and symmetric equations of the line ℓ that passes:

a) Through the point $A(1,3,0)$ and parallel to the vector $\vec{u} = 2i - 5j + 7k$

b) Through the points $A(2,-1,1)$ *and* $B(1,3,-2)$

c) Through the point $A(2,3,-5)$ and parallel to the vector $\vec{u} = 4j + 8k$

Solution: In this problem, the direction vector of the line is given directly in part (a) but in part (b), we determine it from the given points.

a) Vector equation (VE) : $l: X=(1,3,0)+t(2,-5,7)$

Parametric Equations (PE): $\begin{cases} x=1+2t \\ y=3-5t \\ z=7t \end{cases}$ and SE : $\frac{x-1}{2}=\frac{y-3}{-5}=\frac{z}{7}$

b) VE : $l: X=(2,-1,1)+t(-1,4,-3)$, PE : $\begin{cases} x=2-t \\ y=-1+4t \\ z=1-3t \end{cases}$, SE : $\frac{x-2}{-1}=\frac{y+1}{4}=\frac{z-1}{-3}$

c) VE : $l: X=(2,3,-5)+t(0,4,8)$, PE : $\begin{cases} x=2 \\ y=3+4t \\ z=-5+8t \end{cases}$, SE : $x=2; \frac{y-3}{4}=\frac{z+5}{8}$

Note that the vector and or the parametric equations of a line are not unique. That means we can find different parametric equations for the same line.

Definition (Parallel and Perpendicular Lines in Space):

Suppose $l: X=P+t\vec{u}, t\in R$ and $m: X=Q+r\vec{v}, r\in R$ are any two lines. Then, they are said to be

a) *Parallel* if their direction vectors $\vec{u}$ and $\vec{v}$ are parallel. i.e $l // m \Leftrightarrow \vec{u} // \vec{v}$

b) *Perpendicular* if $\vec{u}$ and $\vec{v}$ are perpendicular. i.e $l \perp m \Leftrightarrow \vec{u} \perp \vec{v}$.

Besides, the angle between these lines is the same as the angle between their direction vectors.

Examples:

1. Determine whether the following pairs of lines are parallel, perpendicular or neither and for these which are neither find the angle between them.

a) $l: X=(1,2,3)+t(1,2,-1)$ and $m: X=(1,0,1)+t(-3,-6,3)$

b) The lines through $A=(1,3,5), B=(4,7,5)$ and $C=(5,-2,2), D=(1,1,7)$

c) The lines through $A=(2,-1,4), B=(2,-2,5)$ and $C=(3,4,3), D=(3,5,3)$

Solution: a) Here, $\vec{u}=(1,2,-1)$, $\vec{v}=(-3,-6,3) \Rightarrow \vec{v}=-3\vec{u} \Rightarrow \vec{u} // \vec{v}$. Hence, from the definition, the lines are also *parallel*.

b) In this case, $\vec{u}=B-A=(3,4,0)$, $\vec{v}=D-C=(-4,3,5) \Rightarrow \vec{v}.\vec{u}=0 \Rightarrow \vec{u} \perp \vec{v}$.

Hence, the lines are also *perpendicular*.

c) $\vec{u} = B - A = (0,-1,1)$, $\vec{v} = D - C = (0,1,0)$. But those vectors are neither parallel nor perpendicular and so are the lines through these points. Let θ be the angle between the lines. Then, $\cos\theta = \frac{\vec{u}.\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}(\frac{-1}{\sqrt{2}}) = \frac{3\pi}{4}$.

2. For what value of k are the line $l: x = 2t, y = 1-3t, z = -2-7t$ and the line $m: X = (2,3,-5) + r(3,k,-3)$ perpendicular?

Solution: The direction vectors of the lines are $\vec{u} = 2i - 3j - 7k, \vec{v} = 3i + kj - 3k$.
Then, the two lines will be perpendicular if their direction vectors are perpendicular. So, $\vec{u} \perp \vec{v} \Rightarrow \vec{u}.\vec{v} = 0 \Rightarrow 6 - 3k + 21 = 0 \Rightarrow -3k = -27 \Rightarrow k = 9$.

3. If the lines $l: X = (2,3,1) + t(-1,3,-2)$ and $k: X = (3,2,-4) + r(a,b,-6)$ are parallel to each other, find the values of the constants a and b.

Solution: Two line are parallel if and only if their direction vectors are parallel. Here, the direction vectors of the lines are $u = -i + 3j - 2k$ and $v = ai + bj - 6k$.
But if two vectors are parallel, then one is a scalar multiple of the other.

$l // k \Rightarrow u // v \Rightarrow \lambda u = v \Rightarrow -\lambda i + 3\lambda j - 2\lambda k = ai + bj - 6k$

$$\Rightarrow -\lambda = a, 3\lambda = b, -2\lambda = -6 \Rightarrow \lambda = 3, a = -3, b = 9$$

Intersection of Lines in Space:

Step-1: Write the equations of each lines in parametric form (if they are not).

Suppose $l: \begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct \end{cases}$ and $m: \begin{cases} x = x_1 + mr \\ y = y_1 + nr \\ z = z_1 + pr \end{cases}$ are parametric equations.

Step-2: Equate the parametric equations and solve for the parameters t and r.
Here, a point $P(x,y,z)$ will be the intersection of the lines if and only if it satisfies the equations of both lines at the same time.

Hence, at the point of intersection, we have, $\begin{cases} x_0 + at = x_1 + mr \\ y_0 + bt = y_1 + nr \\ z_0 + ct = z_1 + pr \end{cases}$.

From this simultaneous equation, determine the parameters t and r.

Step-3: Put the values of t and r in each line to find the points P and Q. If $P=Q$, the lines intersect and if $P \neq Q$, the lines never intersect.

Note: Two lines in space that are neither parallel nor intersecting are called *Skew-lines*.

Examples:

a) Given the lines $l: x+1=4t, y-3=t, z-1=0$ and $m: x+13=12r, y-1=6r, z-2=3r$.

Show that the lines are intersecting and find the point of intersection.

b) Find the intersection of the lines $l: x=1+t, y=2+2t, z=9-3t$ and $m: x=4-2r, y=-2+6r, z=1+5r$.

c) Determine whether the lines $l: x=2-t, y=1+2t, z=5+2t$ and $m: x-1=2-y=\frac{z-1}{3}$ are parallel, intersecting, or skew.

Solution: For clarity, let's follow the above three steps.

a) To find the intersection point (if any) of the two lines:

Step-1: Write the equations in parametric form (if they are not).

In this example, the lines are already given in parametric form.

Step-2: Equate the parametric equations and solve for the parameters t and r.

That is $\begin{cases} -1+4t=-13+12r \\ 3+t=1+6r \\ 1=2+3r \end{cases} \Rightarrow r=-\frac{1}{3}, t=-4$

Step-3: Put the values of t and r in each line to find the points P and Q.

Hence, if we put $t=-4$ in the equation of l, we get $P=(-17,-1,1)$.

If we put $r=-\frac{1}{3}$ in the equation of m, we get the point $Q=(-17,-1,1)$.

Here, $P=Q=(-17,-1,1)$ and thus the two line intersect at $(-17,-1,1)$.

b) If the two lines intersect, we have

$$\begin{cases} 1+t=4-2r \\ 2+2t=-2+6r \Rightarrow 2+2(3-2r)=-2+6r \Rightarrow r=1, t=1 \\ 9-3t=1+5r \end{cases}$$

Putting $t=1$ and $r=1$ in the equations of the lines l and m respectively gives the same point $x=2,\ y=4,\ z=6$ and thus the intersection is $(x,y,z)=(2,4,6)$.

c) If the two lines intersect, we have

$$\begin{cases} 2-t=1+r \\ 1+2t=2-r \\ 5+2t=1+3r \end{cases} \Rightarrow 1+2(1-r)=2-r \Rightarrow r=1, t=0$$

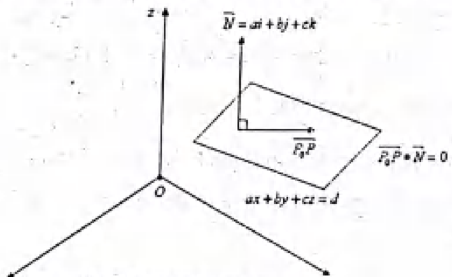
Putting $t=0$ and $r=1$ in the equations of the lines l and m respectively gives the points $x=2,\ y=1,\ z=5$ and $x=2,\ y=1,\ z=4$ which are different.

Thus the lines are not intersecting. Now, let's determine whether they are parallel or not. The direction vectors of the lines are $\vec{u}=-i+2j+2k$ and $\vec{v}=i-j+3k$ which are not parallel and thus the lines themselves are not parallel. This means that the given lines are neither parallel nor intersecting. Hence, they are skew-lines.

2-y = t
y = 2-x
3x = z-1
z = 3t+1

Planes in Space

A vector is said to be normal to a plane if it is perpendicular to all vectors in the plane. That means for any two points A and B lying in the plane, the vector $\overrightarrow{AB}$ is always perpendicular to the normal vector of the plane. Suppose π is a plane passing through the point P_0 and normal to the vector $\overrightarrow{N} = ai + bj + ck$ as shown in the diagram below. Let $P(x, y, z)$ be any point in π.



Normal Equation of a plane

Since $\overrightarrow{N} = ai + bj + ck$ is normal vector of the plane and the points $P_0(x_0, y_0, z_0)$ and $P = (x, y, z)$ are on the plane, the vectors $\overrightarrow{P_0P} = (x - x_0, y - y_0, z - z_0)$ and $\overrightarrow{N} = ai + bj + ck$ are perpendicular. Thus, we have

$$\overrightarrow{P_0P}.\overrightarrow{N} = 0 \Rightarrow (x - x_0, y - y_0, z - z_0).(ai + bj + ck) = 0$$
$$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
$$\Rightarrow ax + by + cz = d, \text{ where } d = ax_0 + by_0 + cz_0$$

Example: Find the equation of a plane passing through the point $(6,1,-3)$ and normal to the vector $\overrightarrow{N} = 3i - 2j + 4k$.

Solution: Take $P_0 = (6,1,-3)$ and let $P = (x, y, z)$ be arbitrary point on the plane. Then, from the definition, we have

$\pi : \overrightarrow{P_0P}.\overrightarrow{N} = 0 \Rightarrow 3(x-6) - 2(y-1) + 4(z+3) = 0 \Rightarrow 3x - 2y + 4z = 4$.

Remark: In order to write the equation of a plane, we need at least one point lying in the plane and a vector normal to the plane. But, sometimes the normal vector may not be given explicitly.

In such cases, we have to use basic concepts of vectors (like cross product) in order to determine a normal vector from the given data in the problem.

Example: Find the equation of a plane through $P(1,2,-1)$, $Q(3,2,4)$ and $R(0,1,5)$.

Solution: Here, three points are given rather than a point and a normal vector. Since the three points are on the plane, the vectors

$\overrightarrow{PQ} = 2i + 5k$, $\overrightarrow{PR} = -i - j + 6k$ are lying on the plane. But from the property of cross product, the vector $\overline{N} = \overline{PQ} \times \overline{PR}$ is perpendicular to both of the vectors and so is to the plane containing them. Hence, we can use this vector as a normal vector and any one of the points to determine the equation of the plane uniquely.

So, $\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 2 & 0 & 5 \\ -1 & -1 & 6 \end{vmatrix} = 5i - 17j - 2k$. Hence, using point $P(1,2,-1)$,

we have $\pi: 5(x-1) - 17(y-2) - 2(z+1) = 0 \Rightarrow 5x - 17y - 2z + 27 = 0$.

Parallel and Perpendicular Planes:

Any two planes in space are said to be

i) parallel if their normals are parallel

ii) perpendicular if their normals are perpendicular. Besides, the angle between any two planes is the same as the angle between their normals.

Examples:

1. Determine whether the pairs of planes are parallel, perpendicular or neither and if they are neither parallel nor perpendicular find the angle between them.

   a) $2x - 3y + 5z - 11 = 0$ and $x + 4y + 2z + 7 = 0$

   b) $2x - 6y + z + 2 = 0$ and $-x + 3y - \frac{1}{2}z - 5 = 0$

   c) $x + y + z - 7 = 0$ and $x + y - z + 13 = 0$

Solutions:

a) $n_1 = 2i - 3j + 5k$, $n_2 = i + 4j + 2k \Rightarrow n_1.n_2 = 2 - 12 + 10 = 0 \Rightarrow n_1 \perp n_2$

Hence, the planes are perpendicular.

b) $n_1 = 2i - 6j + k$, $n_2 = -i + 3j - \frac{1}{2}k \Rightarrow n_1 = -2(-i + 3j - \frac{1}{2}) = -2n_2 \Rightarrow n_1 // n_2$.

This means the plane themselves are parallel.

c) In this case, the normals $n_1 = i + j + k$, $n_2 = i + j - k$ which are neither parallel nor perpendicular and so are the planes. Therefore, the angle between the planes is given by $\cos\theta = \frac{n_1 . n_2}{\|n_1\|\|n_2\|} = \frac{1}{3} \Rightarrow \theta = \cos^{-1}\frac{1}{3}$

2. Find the equation of the plane

a) through $P(-2,1,4)$, $Q(1,0,3)$ and perpendicular to the plane $\pi: 4x - y + 3z = 2$

b) Containing the point $P(-1,2,-5)$ and perpendicular to the planes $\pi_1: 2x - y + z = 1$ and $\pi_2: x + y - 2z = 3$.

Solution:

a) If two planes are perpendicular their normals are perpendicular and thus the normal vector of one plane is parallel to the other plane. Thus, the vector $N_1 = 4i - j + 3k$ is parallel to the required plane. Besides, the plane contains the points $P(-2,1,4)$, $Q(1,0,3)$ and hence the vector $\overline{PQ} = 3i - j - k$ is also parallel to the plane. Hence, its normal is obtained from their cross product as follow:

$$N_2 = N_1 \times \overline{PQ} = \begin{vmatrix} i & j & k \\ 4 & -1 & 3 \\ 3 & -1 & -1 \end{vmatrix} = 4i + 13j - k$$. Hence, the equation of the plane is

$4(x+2) + 13(y-1) - (z-4) = 0 \Rightarrow 4x + 13y - z - 1 = 0$.

b) The normal vectors of the given planes are $N = 2i - j + k$ and $M = i + j - 2k$. But the plane that we are required is given to be perpendicular to the given planes and thus it is parallel to these normal vectors. If it is parallel to the normal vectors, it is perpendicular to their cross product. Hence, the vector $N \times M = i + 5j + 3k$ can be used as a normal vector to the required plane. Thus,

$\pi: x + 1 + 5(y-2) + 3(z+5) = 0 \Rightarrow x + 5y + 3z + 6 = 0$.

3. Find a so that the angle between the planes $ax + y + z = 1$ and $x + z = 1$ is $\theta = \pi/4$.

Solution: The angle between the planes means the angle between their normals.

So, $n_1 . n_2 = \|n_1\|\|n_2\|\cos\theta \Rightarrow a + 1 = \sqrt{a^2 + 2} \Rightarrow 2a + 1 = 2 \Rightarrow a = \frac{1}{2}$.

Relation of a line and a plane in space: Suppose $l: X = A + t\vec{u}$ is a line and π is a plane with normal vector N. Then, the line and the plane are said to be

i) Parallel if and only if $\vec{u}$ and N are perpendicular. That means a line and a plane are parallel if and only if the direction vector of the line is perpendicular to the normal vector of the plane.

ii) Perpendicular if and only if $\vec{u}$ and N are parallel.
That means a line and a plane are perpendicular if and only if the direction vector of the line is parallel to the normal vector of the plane.

Examples:

1. Determine whether the line and the plane are parallel or perpendicular or neither.

a) $l: \frac{x}{3} = y+1 = \frac{z+3}{2}; \pi: x - y - z - 2 = 0$

b) $l: \frac{x-2}{-1} = \frac{y+1}{2} = \frac{z-1}{3}; \pi: 2x - 4y - 6z + 17 = 0$

c) $l: x = 4 + 2t, y = -t, z = -1 - 4t$ and $\pi: 3x + 2y + z - 7 = 0$.

d) $l: x = -1 + 2t, y = 4 + t, z = 1 - t$ and $\pi: 4x + 2y - 2z = 7$.

Solution: Let $\vec{u}$ and N be the direction vector of the line and the normal vector of the plane respectively. Then,

a) $\vec{u} = 3i + j + 2k$, $N = i - j - k \Rightarrow \vec{u}.N = 3 - 1 - 2 = 0 \Rightarrow \vec{u} \perp N \Rightarrow l // \pi$

b) $\vec{u} = -i + 2j + 3k$, $N = 2i - 4j - 6k \Rightarrow N = -2\vec{u} \Rightarrow \vec{u} // N \Rightarrow l \perp \pi$

c) The direction vector of the line $\vec{u} = 2i - j - 4k$ and the normal vector $N = 3i + 2j + k$ are perpendicular and thus the line and the plane are parallel.

d) The direction vector $\vec{u} = 2i + j - k$ and the normal $N = 4i + 2j - 2k$ are parallel and thus the line and the plane are perpendicular.

2. Given $l: x = 4 + 6t, y = 2 - 3t, z = -1 + 9t; \pi: ax - 2y + 6z = 5$. For what value of a do the line and the plane will be Parallel? Perpendicular?

Solution: Here, $\vec{u} = 6i - 3j + 9k$, $N = ai - 2j + 6k$. Then, the line and the plane will be parallel if and only if $\vec{u} \perp N \Leftrightarrow \vec{u}.N = 0 \Leftrightarrow 6a + 6 + 54 = 0 \Leftrightarrow a = -10$.

On the other hand, the line and the plane will be perpendicular if and only if $\vec{u} // N$. But, $\vec{u} // N \Rightarrow \vec{u} = tN \Rightarrow 6 = at, -3 = -2t, 9 = 6t \Rightarrow t = \frac{3}{2} \Rightarrow a = 4$.

3. Find the equation of a line

a) Passing through $(4,1,6)$ and perpendicular to the lines

$l: x = t, y = t, z = t; m: x = 1 - 2t, y = 2 + t, z = -5t$.

b) passing through $(1,-1,0)$ and perpendicular to the plane containing the lines

$l: x = 1 + 3t, y = -1 + 2t, z = 4t; m: x = -4 + 6t, y = 6 + 4t, z = 10 + 8t$

Solution: a) Since the required line is perpendicular to the two given lines, it is also perpendicular to their direction vectors. So, its direction vector will be the cross product of the direction vectors of the given lines.

That is $\vec{u} = \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ -2 & 1 & -5 \end{vmatrix} = -6i + 3j + 3k$.

Hence, its equation is given by $l: x = 4 - 6t, y = 1 + 3t, z = 6 + 3t$.

b) What is the difference between this problem with part (a)? The difference is that in part(a), the two lines are not parallel but in this case the lines are parallel and thus we cannot use the cross product of their direction vectors as the direction vector of the required line. So, what? In this case, let's find another vector parallel to the plane containing these lines but not parallel with the given lines. To find such vector, take one point from each line and form a vector. If we set, $t = 0$, in these lines we get the points $P = (1,-1,0)$ and $Q = (-4,6,10)$.

Hence, the vector $\overrightarrow{QP} = P - Q = 5i - 7j - 10k$ (you can also use $\overrightarrow{PQ}$) is parallel to the plane containing these lines but not with the lines. Here, why we do this is to find two vectors parallel to the plane so that their cross product will be the direction vector of the required line. Thus, $\vec{u} = \begin{vmatrix} i & j & k \\ 3 & 2 & 4 \\ 5 & -7 & -10 \end{vmatrix} = 8i + 50j - 31k$.

Hence, the equation of the line becomes $l: x = 1 + 8t, y = -1 + 50t, z = -31t$.

4. Find the equation of a line m through the point $P(5,9,3)$ and perpendicular to the line $l: \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and determine the point of intersection.

Solution: Any point $Q(x,y,z)$ on the given line is of the form

$Q(x,y,z) = (1+2t, 2+3t, 3+4t)$. Then, $\overrightarrow{PQ} = (2t-4, 3t-7, 4t)$. Now chose Q in such a way that $\overrightarrow{PQ} \perp l$. Then, $\overrightarrow{PQ}$ is also perpendicular to the direction vector of the line. That is $2(2t-4)+3(3t-7)+16t = 0 \Rightarrow t = 1$.

So, the point of intersection is $Q(x,y,z) = (1+2t, 2+3t, 3+4t) = (3,5,7)$.

Hence, the vector $\overrightarrow{PQ} = (2,4,-4)$ is the direction vector of the required line.

Therefore, $m: \frac{x-5}{2} = \frac{y-9}{4} = \frac{z-3}{-4}$ is the required line.

5. Find the equation of a plane

a) Containing the point $P(1,2,-2)$ and perpendicular to $l: \frac{x-1}{2} = \frac{y+4}{3} = z-2$

b) Containing the point $(1,-1,-2)$ and the line $l: \frac{x}{3} = y+1 = \frac{z+3}{2}$

c) Containing the line $l: x = -2+3t, y = 4+2t, z = 3-t$ and perpendicular to the plane $\pi: x-2y+z = 5$.

Solution:

a) Since required plane is perpendicular to the line l, it is also perpendicular to the direction vector $\vec{u} = 2i+3j+k$ of the line.

Thus, by taking this vector as normal vector and the given point $P(1,2,-2)$, we have $\pi: 2(x-1)+3(y-2)+z+2 = 0 \Rightarrow 2x+3y+z = 6$.

b) To find the equation of a plane, whenever a point P and a line l are given take one point, say Q, from the given line and form the vector $\overrightarrow{PQ}$.

Since the line l and the vector $\overrightarrow{PQ}$ are on the plane, the direction vector $\vec{u}$ of the line and the vector $\overrightarrow{PQ}$ are both parallel to the plane and thus their cross product $N = \overrightarrow{PQ} \times \vec{u}$ is perpendicular to the plane and it can be used as normal vector to the plane. So, let $Q(0,-1,-3)$ and $\overrightarrow{PQ} = -i-k$.

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Hence, $N = \overrightarrow{PQ} \times \vec{u} = \begin{vmatrix} i & j & k \\ -1 & 0 & -1 \\ 3 & 1 & 2 \end{vmatrix} = i - j - k$ is normal to the required plane.

Therefore, the equation is $(x-1)-(y+1)-(z+2)=0 \Rightarrow x-y-z-4=0$.

c) If two planes are perpendicular, then the normal vector of one of the plane is parallel to the other plane. So, the normal vector $n = i - 2j + k$ of the given plane is parallel to the required plane. Besides, since the plane contains the line l, the direction vector $\vec{u} = 3i + 2j - k$ of the line is parallel to the plane. Thus, the cross product of these vectors is perpendicular to the plane and it is used as a normal vector. That is the vector $N = n \times \vec{u} = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 2 & -1 \end{vmatrix} = 4j + 8k$ is the normal vector. Here, we have got a normal vector but we also need a point on the plane to give the equation of the plane. How can we get a point? Since the plane contains the line l, we can pick any convenient point from the line.

Therefore, by taking the point $(-2,4,3)$, the equation of the plane becomes

$$4(y-4)+8(z-3)=0 \Rightarrow y+2z-10=0.$$

Intersection of a line and a plane: ⌐ – – – (✱)

Suppose $\pi : ax+by+cz=d$ is a plane and l is a line given in parametric form as $l: x=x_0+at,\ y=y_0+bt,\ z=z_0+ct$. Then, to find the intersection of the line and the plane is a point (x,y,z) which satisfies the equations of the line and the plane at the same time. Thus, to obtain the intersection of the line and the plane, first find the appropriate t by putting the parametric equations of the line in the equation of the plane and then solve for the coordinates (x,y,z) using the value of t. But if you get a contradiction while solving for the parameter t, it means that the lines and the plane never intersect. If you get an equation of the form $0=0$ **or** $a=a,\ a\in R$, it means that the line lies on the plane.

Examples: Find the intersection point of

a) The line $l: x=2-3t, y=4+3t, z=3-5t$ and the plane $\pi: 2x+3y+4z+6=0$

b) The line $l: x=\dfrac{y+1}{2}=\dfrac{z-2}{-3}$ and the plane $x+y+z=1$

c) The line through $P(1,2,0)$, $Q(3,-4,2)$ and the plane $\pi: x+y+2z+9=0$.

Solution:

a) Putting $x=2-3t, y=4+3t, z=3-5t$ in $\pi: 2x+3y+4z+6=0$ gives
$2x+3y+4z+6=0 \Rightarrow 2(2-3t)+3(4+3t)+4(3-5t)+6=0 \Rightarrow 17t=34 \Rightarrow t=2$

Then, putting $t=2$ in the equation of the line gives $x=-4, y=10, z=-7$.

Hence, the intersection point is $(-4,10,-7)$.

b) In this case, the equation of the line is not parametric form. So, first change it in parametric form. The parametric equation becomes

$x=t, y=-1+2t, z=2-3t$. Then, we have

$x+y+z=1 \Rightarrow t-1+2t+2-3t=1 \Rightarrow 1=1$ which is true for all values of t.

Therefore, the line lies in the plane.

c) Here, first find the parametric equation of the line through $P(1,2,0)$, $Q(3,-4,2)$ which is $x=1+2t, y=2-6t,\ z=2t$. Then,

$x+y+2z+9=0 \Rightarrow 1+2t+2-6t+4t+9=0 \Rightarrow 12=0$ which is a contradiction.

Therefore, the line and the plane never intersect.

Intersection of Planes in space:

If two planes intersect, their intersection is always a line. So, to find the intersection of two given planes, first eliminate one of the variables from the equation of the planes, second assign a parameter t for one of the two remaining variables and then solve for the other variable in terms of the parameter. Finally, solve for the variable eliminated in the first step. This will give you the parametric equations of the intersection of the two planes. Generally, if π_1 and π_2 are two intersecting planes with normal N_1 and N_2, their line of intersection will have a direction vector $\vec{u} = N_1 \times N_2$.

Examples:

1. Find the equation of a line through

a) $(0,0,0)$ parallel to the line of intersection of the planes $2x-3y+4z=2$ and $x-z=1$.

b) $P(5,0,-2)$ that is parallel to $\pi_1 : x-4y+2z=0$ and $\pi_2 : 2x+3y-z+1=0$.

Solution:

a) Since the intersection line is contained by both planes, the normal vectors of these planes are both perpendicular to the line of intersection. Thus, their cross product becomes parallel to the required line.

Hence, the direction vector is $\vec{u} = n_1 \times n_2 = \begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ 1 & 0 & -1 \end{vmatrix} = 3i+6j+3k$. Hence the required line is $l : X = (0,0,0)+t(3,6,3)$ or $l : x=3t, y=6t, z=3t$.

b) A line is parallel to a plane if and only if the direction vector of the line is perpendicular to the normal vector of the plane. In our case, the required line is perpendicular to the normal vectors of the planes and thus it is parallel to their cross product. Hence, the direction vector of the line is

$\vec{u} = N_1 \times N_2 = \begin{vmatrix} i & j & k \\ 1 & -4 & 2 \\ 2 & 3 & -1 \end{vmatrix} = -2i+5j+11k$. Hence, the equation of the line is

$l : x = 5-2t, y = 5t, z = -2+11t$.

2. Find the equation of the plane through $P(-1,4,2)$ and containing the line of intersection of the planes $4x-y+z=2$ and $2x+y-2z=3$.

Solution: First, find the parametric equation of the intersection line of the given planes. The intersection line is obtained by solving the equation $\begin{cases}4x-y+z=2\\2x+y-2z=3\end{cases}$

Subtracting two times the second from the first and solving for x, y in terms of $z=t$, gives $x=\frac{1}{6}t+\frac{5}{6}$, $y=\frac{5}{3}t+\frac{4}{3}$. Now take one point from this line say $Q(1,3,1)$ (letting $t=1$). Since the plane contains the intersection line and the points P and Q, the direction vector $\vec{u}=\frac{1}{6}i+\frac{5}{3}j+k$ of the line and the vector $\overline{PQ}=2i-j-k$ are parallel to the required plane and thus their cross product

$$N=\vec{u}\times\overline{PQ}=\begin{vmatrix}i & j & k\\ \frac{1}{6} & \frac{5}{3} & 1\\ 2 & -1 & -1\end{vmatrix}=-\frac{2}{3}i+\frac{13}{6}j-\frac{21}{6}k$$ is normal to the plane.

Therefore, by using the given point $(-1,4,2)$, the equation of the plane becomes

$$-\frac{2}{3}(x+1)+\frac{13}{6}(y-4)-\frac{21}{6}(z-2)=0\Rightarrow 4x-13y+21z+14=0$$

3. Find the equation of a plane containing the lines

a) $l: x=1+3t, y=1-t, z=2+t$ and $m: x=1+4t, y=-1+2t, z=5+t$

b) $l: x=1+t, y=1+2t, z=3+t$ and $m: x=3+t, y=2t, z=-2+t$

Solution:

a) Since the required plan contains the lines , its normal vector is perpendicular to the two given lines, it is also perpendicular to their direction vectors. So, the normal vector will be the cross product of the direction vectors of the lines. That is $\vec{u}=u_1\times u_2=-3i+j+10k$.

Now, let's take one a point from one of the lines, $P=(1,1,2)$ from the first line by letting $t=0$, you can also take from the second line, any point by giving different values for t always the answer is the same. Hence, the equation of the plane becomes $-3(x-1)+(y-1)+10(z-2)=0\Rightarrow -3x+y+10z-18=0$.

b) What is the difference between this problem with part (a)? The difference is that in part(a), the two lines are not parallel but in this case the lines are parallel and thus we cannot use the cross product of their direction vectors as the normal vector of the required plane. In this case, let's find another vector parallel to the plane containing these lines but not parallel with the given lines. To find such vector, take one point from each line and form a vector. If we set, $t=0$, in these lines we get the points $P=(1,1,3)$ and $Q=(3,0,-2)$.

Hence, the vector $\overrightarrow{PQ}=2i-j-5k$ is parallel to the plane containing these lines but not with the lines. Thus, $\vec{N}=\begin{vmatrix} i & j & k \\ 2 & -1 & -5 \\ 1 & 2 & 1 \end{vmatrix}=9i-7j+5k$. Hence, by using either of the points on the lines, the equation of the line becomes

$$9(x-1)-7(y-1)+5(z-3)=0 \Rightarrow 9x-7y+5z-17=0.$$

4. Given the lines $l: x-2=\dfrac{y-4}{2}=\dfrac{z}{-3}$ and $m:\begin{cases} x=r, \\ y=2r \\ z=-r \end{cases}$

a) Find the intersection point of the lines.

b) Write the equation of the plane determined by these lines.

Solution: Here, the two lines are expressed in parametric equations as

$$l:\begin{cases} x=2+t, \\ y=4+2t \\ z=-3t \end{cases} \quad \text{and} \quad m:\begin{cases} x=r, \\ y=2r \\ z=-r \end{cases}$$

a) **First:** Equate the parametric equations of the two lines to get the parameters t and r. That is $l:\begin{cases} 2+t=r \\ 4+2t=2r \\ -3t=-r \Rightarrow r=3t \end{cases} \Rightarrow t=1, r=3$

Second: Determine two points P and Q on the lines using $t=1$ and $r=3$ in the equations of the lines. That is

Using $t=1$ in l, $P=(3,6,-3)$ and using $r=3$ in m, we get $Q=(3,6,-3)$

Since $P=Q=(3,6,-3)$, the two line intersect at $P=(3,6,-3)$

6. Given the lines $l: \frac{x}{3} = \frac{y}{-2} = \frac{z}{4}$ and $m: x = 4 - k, y = 4 + 2k, z = -3 - 3k$.

a) Find the intersection point of the lines (if any)

b) Write the equation of a plane containing the two lines.

c) Give the cosine value of the angle between the two lines.

Solution:

a) **Intersection of lines:** To find intersection of any two lines in space:

First: Express each line in parametric equation (if they are not)

Second: Equate the parametric equations and solve for the parameters.

Third: Put the parameters in each line to obtain two points P and Q.

Decision: If the two points P and Q are the same, the lines intersect at that point.

Now, let's use the above steps to make the idea clear.

First: Express each line in parametric equation (if they are not)

That is $l: x = 3t, y = -2t, z = 4t$ and $m: x = 4 - k, y = 4 + 2k, z = -3 - 3k$

Second: Equate the parametric equations and solve for the parameters t and k

That is $\begin{cases} 3t = 4 - k \\ -2t = 4 + 2k \\ 4t = -3 - 3k \end{cases} \Rightarrow \begin{cases} 6t = 8 - 2k \\ -2t = 4 + 2k \end{cases} \Rightarrow 4t = 12 \Rightarrow t = 3,\ k = -5$

Third: Put the parameters in each line to obtain two points P and Q.

Putting $t = 3$ in l and $k = -5$ in m, we get $P = (9, -6, 12)$ and $Q = (9, -6, 12)$.

Decision: Since $P = Q$, the lines intersect at $P = (9, -6, 12)$.

b) **Equation of a plane:** If a plane containing two intersecting lines is required, its normal vector is the cross product of the direction vectors of the lines. Here, the direction vectors are $\vec{u} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ for l and $\vec{v} = -\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ for m.

Hence, the normal vector is $\mathbf{N} = \vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ -1 & 2 & -3 \end{vmatrix} = -2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$

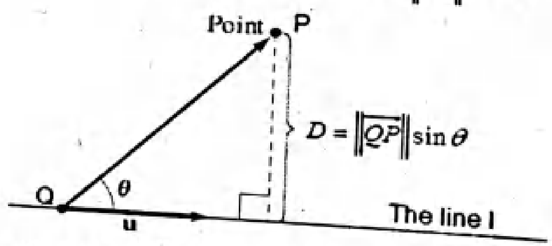
Equation: Once the normal vector is found, we can use the intersection point or we can select any other point from one of the lines to write the equation.

Hence, $-2(x - 9) + 5(y + 6) + 4(z - 12) = 0 \Rightarrow 2x - 5y - 4z = 0$

2.7.3 Computation of Distance

In many situations, we need to compute distances between different locations or sites. But computing distance between different objects is not an easy task. One of the applications of vectors is to simplify computation of distances in space.

Distance from a point to a line: Let l be a line given by $l: X = Q + t\vec{u}$. If P is any point not on the given line l as shown in the diagram below, then the shortest distance from the line to the point P is given by $D = \|\overrightarrow{QP}\|\sin\theta$.



The distance between a point and a line

But $D = \|\overrightarrow{QP}\|\sin\theta = \|\overrightarrow{QP}\|\left(\frac{\|\overrightarrow{QP}\times u\|}{\|\overrightarrow{QP}\|\|u\|}\right) = \frac{\|\overrightarrow{QP}\times u\|}{\|u\|}$, $\therefore \sin\theta = \frac{\|\overrightarrow{QP}\times u\|}{\|\overrightarrow{QP}\|\|u\|}$

In general, the distance between a point P and any line l is given by

$D = \frac{\|\overrightarrow{QP}\times u\|}{\|u\|}$ where Q is arbitrary point taken from the given line. Different students can take different values for Q but the final answer is the same.

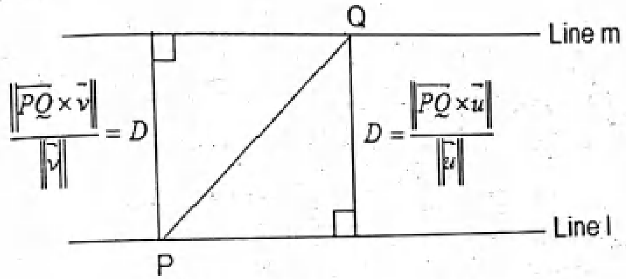
Example: Find the distance between the point $P(1,-2,0)$ and the line given by $l: x = 2+3t,\ y = -2-3t,\ z = -t$.

Solution: Here, the direction vector of the line is $\vec{u} = 3i - 3j - k$ and we can take infinitely many points from the given line by letting different values for the parameter t. Thus, by letting $t = 0$, we get a point $Q(2,-2,0)$ on the line such that $\overrightarrow{QP} = -i$.

Thus, $\|\overrightarrow{QP}\times\vec{u}\| = \begin{vmatrix} i & j & k \\ -1 & 0 & 0 \\ 3 & -3 & -1 \end{vmatrix} = -j+3k$. Hence, $D = \dfrac{\|\overrightarrow{QP}\times\vec{u}\|}{\|\vec{u}\|} = \dfrac{\sqrt{10}}{\sqrt{19}} = \sqrt{\dfrac{10}{19}}$

Distance between two lines:

Case I: When the lines are parallel: The above formula is also applicable to find the distance between two parallel lines. Consider the diagram where the lines are parallel.



The distance between two parallel lines

That is if $l: X = P + t\vec{u}$ and $m: X = Q + r\vec{v}$ are any two parallel lines as shown in the diagram, then the distance between them is given by $D = \dfrac{\|\overrightarrow{PQ}\times\vec{u}\|}{\|\vec{u}\|}$ or

$D = \dfrac{\|\overrightarrow{PQ}\times\vec{v}\|}{\|\vec{v}\|}$ where P and Q are points taken one from each line.

Example: Find the distance between the lines $l: x = 1+t, y = -1+2t, z = 3-t$ and $m: x = 2-t, y = 1-2t, z = t$

Solution: Here, $\vec{u} = i+2j-k$, $\vec{v} = -i-2j+k \Rightarrow \vec{u} = -\vec{v}$ and thus the lines are parallel. So, take $P(1,-1,3)$ from l and $Q(2,1,0)$ from m such that

$\overrightarrow{PQ} = i+2j-3k$.

Then, $\overrightarrow{PQ}\times\vec{u}=\begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ 1 & 2 & -1 \end{vmatrix}=4i-2j$. This gives, $D=\dfrac{\|\overrightarrow{PQ}\times\vec{u}\|}{\|\vec{u}\|}=\dfrac{\sqrt{20}}{\sqrt{6}}=\sqrt{\dfrac{10}{3}}$.

Case II: When the lines are skew: Suppose the lines $l: X=P+t\vec{u}$ and $m: X=Q+r\vec{v}$ are skew lines. To determine the distance between these lines, select a point P from l and a point Q from m (the selection of these points is arbitrary, you can select any point you wish).

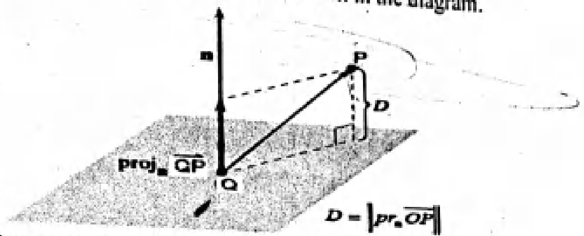
Then, the distance between the lines is the length of the projection of the vector $\overrightarrow{PQ}$ on a vector perpendicular to both lines. Since $\vec{u}$ and $\vec{v}$ are the direction vectors of the lines, a vector perpendicular to both lines means a vector perpendicular to both $\vec{u}$ and $\vec{v}$. But a vector perpendicular to both $\vec{u}$ and $\vec{v}$ is given by $\vec{w}=\vec{u}\times\vec{v}$. Hence, the distance between the lines is given by $D=\|pr_{\vec{w}}\overrightarrow{PQ}\|=\left\|\left(\dfrac{\overrightarrow{PQ}.\vec{w}}{\|\vec{w}\|^2}\right)\vec{w}\right\|=\dfrac{|\overrightarrow{PQ}.\vec{w}|}{\|\vec{w}\|}=\dfrac{|\overrightarrow{PQ}.(\vec{u}\times\vec{v})|}{\|\vec{u}\times\vec{v}\|}$.

Example: Find the distance between $l: x=1+t, y=2+t, z=5+t$ and $m: x=2+r, y=4+2r, z=9+3r$.

Solution: Here, $\vec{u}=i+j+k$, $\vec{v}=i+2j+3k \Rightarrow \vec{u}\times\vec{v}=\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix}=i-2j+k$.

Clearly, the lines are skew (Verify!). Now, take $P(1,2,5)$ from l and $Q(2,4,9)$ from m such that $\overrightarrow{PQ}=i+2j+4k$. Thus, $D=\dfrac{|\overrightarrow{PQ}.(\vec{u}\times\vec{v})|}{\|\vec{u}\times\vec{v}\|}=\dfrac{|1-4+4|}{\sqrt{6}}=\dfrac{\sqrt{6}}{6}$.

Distance between a point and a plane: Suppose $\pi : ax+by+cz+d=0$ is a plane and $P(x_0, y_0, z_0)$ is a point not on the plane as shown in the diagram.



The distance between a point and a plane

Then, the distance D from the given point to the plane is given by $D = \left\| pr_\mathbf{n} \overrightarrow{QP} \right\|$ where $\mathbf{n} = ai + bj + ck$ is the normal vector of the plane.

So, using $d = -(ax+by+cz)$,

$$D = \left\| pr_\mathbf{n} \overrightarrow{QP} \right\| = \left\| \left(\frac{\overrightarrow{QP}.\mathbf{n}}{\|\mathbf{n}\|^2} \right) \mathbf{.n} \right\| = \left| \left(\frac{\overrightarrow{QP}.\mathbf{n}}{\|\mathbf{n}\|^2} \right) \right| \|\mathbf{n}\| = \left| \left(\frac{\overrightarrow{QP}.\mathbf{n}}{\|\mathbf{n}\|} \right) \right|$$

$$= \frac{\left| \overrightarrow{QP}.\mathbf{n} \right|}{\|\mathbf{n}\|} = \frac{|a(x_0-x)+b(y_0-y)+c(z_0-z|}{\sqrt{a^2+b^2+c^2}}$$

$$= \frac{|ax_0+by_0+cz_0-(ax+by+cz)|}{\sqrt{a^2+b^2+c^2}} = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$

Therefore, the distance between a point $P(x_0, y_0, z_0)$ and the plane $\pi : ax+by+cz+d=0$ is given by $D = \left\| pr_\mathbf{n} \overrightarrow{QP} \right\| = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.

Examples: Find the distance between

a) The point $P(0,2,5)$ and the plane $\pi : 4x-2y-4z+9=0$.

b) The point $P(5,-3,2)$ and the plane $\pi : 3x-4y+12z=12$.

c) The point $P(1,2,3$ and the plane determined by $A(0,1,0)$, $B(2,3,1)$, $C(5,7,2)$.

Solution:

a) Here, the normal vector of the plane is $N = 4i-2j-4k$.

Hence, we have $D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|0-4-20+9|}{\sqrt{16+4+16}}=\left|\frac{-15}{6}\right|=\frac{5}{2}$.

b) Here, the normal vector of the plane is $N=3i-4j+12k$.

Hence, we have $D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|15+12+24-12|}{\sqrt{9+16+144}}=\frac{39}{13}=3$.

c) In this case it is our task to determine the normal vector of the plane first. Since the three points lie on the plane, the vectors $\overrightarrow{AB}=2i+2j+k$, $\overrightarrow{AC}=5i+6j+2k$ are parallel to the plane and their cross product $N=\overrightarrow{AB}\times\overrightarrow{AC}=-2i+j+2k$ is normal to the plane. Hence, by using the point $A(0,1,0)$ (you can also use B or C) as Q in the proof of the theorem, we get the distance,

$$D=\frac{|\overrightarrow{AP}.\mathbf{N}|}{\|\mathbf{N}\|}=\frac{|(i+j+3k).(-2i+j+2k)|}{\sqrt{4+1+4}}=\frac{|5|}{\sqrt{9}}=\frac{5}{3}.$$

Remark: The distance between two planes or a line and a plane can be computed using the formula of the distance between a point and a plane that we have discussed so far.

Examples: Find the distance between the line l and the plane where

a) $l: x=2+6t, y=-1+2t, z=1+3t$ and the plane $\pi: 2x+3y-6z+26=0$.

b) The planes $\pi: 2x+y-2z+4=0$ and $\pi: 4x+2y-4z-19=0$.

Solution:

a) Take a point from the line say $P(2,-1,1)$ by putting $t=0$. Then, apply the distance formula between a point and a plane.

So, $D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|4-3-6+26|}{\sqrt{4+9+36}}=\frac{|21|}{\sqrt{49}}=\frac{21}{7}=3$

$$D=\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}=\frac{|4+3-6+20|}{\sqrt{4+9+36}}=\left|\frac{21}{\sqrt{49}}\right|=\frac{21}{7}=3.$$

b) Take a point from the first plane say $P(0,0,2)$ by putting $x=y=0$ and solving for z. Then, apply the distance formula between a point and a plane.

So, $D=\frac{|0+0-8-19|}{\sqrt{16+4+16}}=\frac{|-27|}{\sqrt{36}}=\frac{27}{6}=\frac{9}{2}=4.5$.

Review Problems on Chapter -2

1. Find the scalar α such that $\alpha\vec{a}+\vec{b}=-3\vec{c}$ where
$\vec{a}=3i+2j-2k, \vec{b}=6i+j+2k, \vec{c}=j-2k$ **Answer** :$\alpha=-2$

2. Let $\vec{a}=i+j-k$, $\vec{b}=2i-j+2k$ and $\vec{c}=2i-j+k$. Then, find

a) $\|2\vec{a}-\vec{b}\|$ b) The projection of $\vec{b}$ onto $\vec{a}$, $pr_{\vec{a}}\vec{b}$

c) The cosine directions of the vector $\vec{a}$

d) A unit vector perpendicular to both $\vec{a}$ and $\vec{b}$

e) The area of a parallelogram determined by $\vec{a}$ and $\vec{b}$

3. The dot products of a certain vector $\vec{v}$ with the vectors $\vec{a}=5k, \vec{b}=3j$ and $\vec{c}=4i$ are 15,6 and 4 respectively, find vector $\vec{v}$. **Answer:** $\vec{v}=i+2j+3k$

4. Given $A=(2,3,0), C=(3,-4,1), D=(5,-4,2)$ and $\|\overrightarrow{AB}\|=5$. If vector $\overrightarrow{AB}$ is parallel to $\mathbf{v}=\overrightarrow{CD}$, find the coordinates of B. **Answer:** $(6,3,2)$ or $(-2,3,-2)$

5. Let $\vec{v}=ai+bj$. If $\|\vec{v}\|=\sqrt{8}$ and $\vec{v}$ is perpendicular to $u=-i+j$, then find $\vec{v}$.

Answer: $\vec{v}=\pm 2(i+j)$

6. For a certain vector $\vec{x}$, $\vec{u}\times\vec{x}=2i+aj-3k$ where $\vec{u}=i+2j+6k$. Then, what must be the value of the constant a? **Answer:** $a=8$.

7. If $\vec{a}\times\vec{b}=i+2j+2k$, $\|\vec{a}\|=\|\vec{b}\|=\sqrt{5}$, then find the cosine value of the angle between the vectors $\vec{a}$ and $\vec{b}$. **Answer:** $\cos\theta=\pm\frac{4}{5}$

8. Given $\|\vec{a}\|=3\sqrt{6}, \|\vec{b}\|=12$ and $pr_{\vec{b}}\vec{a}=2i+4j-4k$. Then, find $\|pr_{\vec{a}}\vec{b}\|$ and give the cosine value of angle θ between $\vec{a}$ and $\vec{b}$. **Answer:** $4\sqrt{6}, \pm\frac{\sqrt{6}}{3}$

9. Suppose $\|\vec{a}\|=2$, $\|\vec{b}\|=13, \vec{a}.\vec{b}=10$. Then, find $\|\vec{a}\times\vec{b}\|$. **Answer:** 24

10. Let $\vec{v}=(2,3,6)$. If $\|\alpha\vec{v}\|=14$, find the value of α. **Answer:** $\alpha=\pm 2$

11. If $\vec{u} = i + j + k$, find vector $\vec{x}$ with the property $\vec{u} \times \vec{x} = i - j$ and $\|\vec{x}\| = \sqrt{2}$.

Answer: $\vec{x} = \frac{1}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{4}{3}\mathbf{k}$ or $\vec{x} = -\mathbf{i} - \mathbf{j}$

12. For a certain vector $\vec{x}$, $\vec{x} // \vec{b}$ where $\vec{b} = 3j - 4k$ and $\|\vec{x}\| = 10$, find vector $\vec{x}$.

Answer: $\vec{x} = 6j + 8k, -6i - 8k$

13. Suppose two vectors $\vec{a}$ and $\vec{b}$, each with norm 6 units, lie in a plane $x + 2y - 2z = 9$. If the angle between them is $\theta = \frac{\pi}{6}$, find the vector $\vec{a} \times \vec{b}$.

Answer: $\vec{a} \times \vec{b} = 6i + 12j - 12k, -6i - 12j + 12k$

14*. If $\|\vec{u}\| = 5, \|\vec{v}\| = 2, \vec{u}.\vec{v} = -6$ and $\vec{w} = \vec{u} \times \vec{v}$, then $\|\vec{u} + \vec{u} + \vec{w}\| =$ ___. **Ans.** 9

15. If $\|\vec{a}\| = 4, \|\vec{b}\| = 10$ and $\|\vec{a} \times \vec{b}\| = 32$, then find $\vec{a}.\vec{b}$. **Answer** : ± 24

16. If $\vec{u}$ and $\vec{v}$ are perpendicular unit vectors, find $\|3\vec{u} - 4\vec{v}\|$. **Answer:** 5

17. Suppose $\vec{a}$ and $\vec{b}$ are unit vectors. Then, find an angle θ such that $\vec{a} - \vec{b}$ is also a unit vector.

Answer: $\frac{\pi}{3}$

18. For what value of $\mathbf{t}$, the four points $A = (1,3,\mathbf{t}), B = (3,-1,6), C = (5,2,0)$ and $D = (3,6,-4)$ are coplanar (they lie in the same plane). **Answer:** $t = 2$

19*. Given $\|\vec{a}\| = 3\sqrt{6}, \|\vec{b}\| = 12, pr_{\vec{b}}\vec{a} = 2i + 4j - 4k$ and the orthogonal projection of $\vec{a}$ onto $\vec{b}$ is $^{\perp}pr_{\vec{a}}\vec{b} = -4i + 4j - 4k$. Find $\vec{a}, \vec{b}$ and $pr_{\vec{a}}\vec{b}$.

Answer: $\vec{a} = \pm(6\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}), \vec{b} = \pm(4\mathbf{i} + 8\mathbf{j} - 8\mathbf{k}), pr_{\vec{a}}\vec{b} = \pm(-8i - 4j + 4k)$

20. Suppose vector $\vec{a}$ is found in the first octant such that $\|\vec{a}\| = 14$ and the direction cosines with respect to x and y axes are $\frac{3}{7}$ and $\frac{2}{7}$ respectively. Find the direction cosine with respect to z axis and give vector $\vec{a}$. **Answer:** $\vec{a} = 6\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$

21. Suppose $\vec{v}$ is a vector in space where $v_x = 3$ and $v_y = 4$. If α, β and γ are its direction angles with $\gamma = 30^0$, find vector $\vec{v}$. **Answer:** $\vec{v} = 3\mathbf{i} + 4\mathbf{j} + 5\sqrt{3}\mathbf{k}$

22*. Find the area of a parallelogram $ABCD$ whose adjacent sides are the vectors $\vec{a}$ and $\vec{b}$ where $\|\vec{a}\| = 6, \|\vec{b}\| = 4$ and $\vec{a}.\vec{b} = 2$.

23. Find the surface area of the rectangular parallelepiped formed by the adjacent vectors $\vec{a} = i + 2j + k, \vec{b} = i - j + k, \vec{c} = i + k$. **Answer:** $12\sqrt{2}$

24. Find the volume of

a) The tetrahedron formed by the points $P(1,1,3), Q(4,3,2), R(5,2,7), S(6,4,8)$

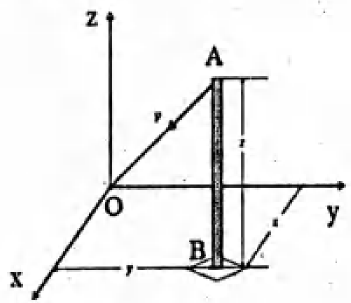
b) The parallelepiped with adjacent sides PQ, PR, PS where $P(1,1,1), Q(2,0,3), R(4,1,7)$ and $S(3,-1,-2)$ **Answer** : *a)* 14/3 *b)* 21

25. The cable AO exerts a tension force $F = -120\mathbf{i} - 80\mathbf{j} - 90\mathbf{k}$ on top of the pole as shown. If the cable is $68m$, then determine the following quantities.

a) The height z of the pole and the coordinates of the point at the base B

b) A unit vector parallel to the force F

c) The projection of the force on the lines AB and OB



26. If the lines $l: X = (2,3,1) + t(-1,3,-2)$ and $k: X = (3,2,-4) + r(a,b,-6)$ are parallel to each other, find the values a and b. **Answer:** $a = -3, b = 9$

27. Find the numbers x and y such that the point $(x, y, 1)$ lies on the line passing through the points $(2,5,7)$ and $(0,3,2)$. **Answer** : $x = -2/5, y = 13/5$

28. Find the point of intersection between

a) the line $x = 3 + 3t, y = 1, z = 4 - 2t, t \in R$ and the plane $4x + 5y - z = -1$.

b) the line $l: X = (1,-1,2) + t(1,1,2)$ and the plane $\pi: 2x - 3y + z - 9 = 0$.

Answer : a) $(0,1,6)$ *b)* $(3,1,6)$

29. Find the distance between:

a) the line $l: x=-1+t, y=3+2t, z=1-t$ and the plane $x-y-z+2=0$.

b) The lines $l: \frac{x-1}{2}=\frac{y+1}{-1}=\frac{z-2}{-2}$ & $m: \frac{x}{2}=\frac{y-2}{-1}=\frac{z-3}{-2}$.

Answer : $a)\ \sqrt{3}$ $\quad b)\ \frac{5\sqrt{2}}{3}$

30*. A plane contains the points $P(-4,9,-9), Q(5,-9,6)$ and is perpendicular to the line through $A(4,-6,k)$. Find the value of k and write the equation of the plane.

Answer : $k=\frac{51}{5}, 5x-15y-21z=34$

31*. Find the equation of the line through $(2,1,0)$ and parallel to the intersection of the planes $x+3y-z=5$ and $2x-2y+4z=2$. **Answer** : $\frac{x-2}{10}=\frac{1-y}{6}=-\frac{z}{8}$

32. Given the lines $l: \frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8}$, $m: x-4=\frac{y+3}{-4}=\frac{z+1}{7}$.

a) Show that the lines are intersecting and determine the point of intersection

b) Find the equation of a plane containing these lines.

Answer : $a)\ (5,-7,6)$ $\quad b)\ 11x-6y-5z=67$

33*. Find the point of intersection of

a) The plane $3x-y+4z=7$ and the line through $(5,4,-3)$ that is perpendicular to the given plane.

b) The line through $(1,-3,1)$ and $(3,-4,2)$ with the plane $x-y+z=2$.

Answer : $a)\ (\frac{77}{13},\frac{48}{13},-\frac{23}{13})$ $\quad b)\ (-\frac{1}{2},-\frac{9}{4},\frac{1}{4})$

34. Find the equation of the plane

a) through $(-2,3,4)$ perpendicular $2x+3y+4z=6$ and $3x+2y+2z=8$

b)* Through $(2,-1,0), (3,-4,5)$ and parallel to the line $l: 2x=3y=4z$.

c) through $(-2,1,7)$ and perpendicular to the line $x-4=2t, y+2=3t, z=-5t$

d) through $(3,-6,7)$ and parallel to the plane $5x-2y+z-5=0$.

Answer : $a)\ 2x-8y+5z+8=0$ $\quad b)\ 29x-27y-22z=85$

$c)\ 2x+3y-5z+36=0$ $\quad d)\ 5x-2y+z-34=0$

35. For what value of the constant **t** does the volume determined by the vectors $\vec{a}=4i+j+2k, \vec{b}=2i+2j-3k$ and $\vec{c}=i+4j+tk$ is zero?

CHAPTER-3

VECTOR SPACES

(Refer "***A Hand Book of Linear Algebra***" written by the same Author for more details about Vector Spaces. This is simply for introduction)

3.1 Definitions and Examples

Let V be an arbitrary nonempty set of objects on which two operations are defined: addition, and multiplication by scalars. By ***addition*** we mean a rule for associating with each pair of objects u and v in V an object $\mathbf{u}+\mathbf{v}$, called the ***sum*** of u and v; by ***scalar multiplication*** we mean a rule for associating with each scalar k and each object u in V an object $k\mathbf{u}$, called the ***scalar multiple*** of u by k. If the following axioms are satisfied by all objects $\mathbf{u}, \mathbf{v}$, in V and all scalars k and m, we call V a ***vector space*** and we call the objects in V **vectors.**

Axioms of Vector Space

1. V is closed under addition: That is $u,\ v \in V \Rightarrow u+v \in V$.
2. Addition is commutative on V : That is $u+v=v+u$.
3. Addition is associative on V : That is $u+(v+w)=(u+v)+w$.
4. There is identity element for addition. Usually, called **zero element** and denoted by $\mathbf{0}$ such that $u+\mathbf{0}=\mathbf{0}+u=u,\ \forall u \in V$.
5. For every u in V , there exists $-u$ in V such that $u+-u=-u+u=\mathbf{0}$. The element $-u$ is called **additive inverse** of u in V .
6. V is closed under scalar multiplication. For u in V and scalar k, ku in V .
7. There exists a unit element, denoted by $\mathbf{1}$, for multiplication such that $\mathbf{1}u=u,\ \forall u \in V$ .This element is known as multiplicative identity or unity.
8. $k(u+v)=ku+kv$.
9. $(k+m)u=ku+mu$.
10. $k(mu)=(km)u$.

Examples of General Vector Spaces: Here, under commonly encountered vector spaces are listed without verification of the definition. Since verifying each of these properties is straight forward, please verify by yourself for your understanding. Furthermore, since we will frequently refer directly or indirectly these vector spaces, you are advised to visualize these spaces by analyzing different examples using the respective operations defined on them.

1. **(Euclidean Vector spaces):** Let $V = R^n$ and consider the usual coordinate wise addition and scalar multiplication. That is for any vectors $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n)$ in R^n and for any real number k,

$$\text{Addition}: x + y = (x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$$

$$\text{Scalar multiplication}: kx = (kx_1, kx_2, ..., kx_n).$$

Then, the set V together with theses operations forms a vector space. Such vector spaces are known as Euclidean vector spaces. You can verify that the zero element of this space is the zero vector $\mathbf{0}$ whose all coordinates are zero. The additive inverse of every vector is its negative multiple. The unity element is the number 1 itself.

2. **(Polynomial Space, P_n):** The set of polynomials of degree less or equal to n are commonly denoted by P_n. This set together with the usual addition and scalar multiplication of polynomials forms a vector space. This vector space is known as Polynomial vector space. In this space, the zero element is the zero polynomial $f(x) = \mathbf{0}, \forall x \in R$.

The additive inverse of every polynomial is its negative multiple. The unity element is the number 1 itself.

3. **(Function space, F):** Let F be the set of all functions with addition and scalar multiplication defined as follow. $(f + g)(x) = f(x) + g(x)$, $(kf)(x) = kf(x)$.

Then, F with these operations forms a vector space.

4. **(Space of Matrices, $M_{m\times n}$):** Let M be the set of all $m \times n$ matrices. Then M together with the usual matrix addition and scalar multiplication forms a vector space. This vector space is known as space of matrices. In this space, the zero elements is the zero matrix $\mathbf{0} = (0_{ij})_{m\times n}$.

Particular examples of vector spaces:

1. Verify that $V = \{0\}$ (the set consisting of only zero element) is a vector space known as *trivial vector space*.

2. Let $V = R^+$ and define addition and scalar multiplication as follow:

$x + y = xy, \ \forall x, y \in R^+$, $k.x = x^k$ where the expression xy denotes the usual multiplication. Show that V together with these operations forms a vector space.

Justification: Let a, b be any two numbers from R^+.

i) **Closure property:** Since x, y are positive and so is their product $x.y$. But $x + y = xy$ which means $x + y$ is also positive and thus in R^+.

ii) **Commutative property:** Since multiplication is commutative, we have $xy = yx \Rightarrow x + y = y + x$.

iii) **Existence of zero:** From the definition,

$x + \mathbf{0} = x.\mathbf{0} \Rightarrow x + \mathbf{0} = x = x\mathbf{0} \Rightarrow \mathbf{0} = 1$. So, the zero element is the number 1 which is in R^+. (Don't be surprise here that 1 is used as zero element because you are going to see more properties that are different from the usual cases)

iv) **Existence of additive inverse:** For any positive real number x, we need to find another positive number y such that $x + y = y + x = \mathbf{0} = 1$.

But, for any positive x, its reciprocal $1/x$ is also positive such that

$$x + \frac{1}{x} = x.\frac{1}{x} = \frac{1}{x}.x = 1 = \mathbf{0}.$$

Hence, the additive inverse of every x is its reciprocal $1/x$ which is also in R^+.

v) **Associative property:** For x, y, z in R^+,

$(x + y) + z = (xy) + z = (xy)z = x(yz) = x + yz = x + (y + z)$

vi) **Closure property under scalar multiple:** For scalar k, $kx = x^k$ but x^k is always positive and thus it is in R^+.

vii) For any scalar k, $k(x + y) = (x + y)^k = (xy)^k = x^k y^k = x^k + y^k = kx + ky$.

(Here, $(xy)^k = x^k y^k$. Why?)

viii) For scalars k, m, $(k + m)x = x^{k+m} = x^k x^m = x^k + x^m = kx + mx$

viiii) For scalars k, m, $k(mx) = k(x^m) = (x^m)^k = x^{mk} = x^{km} = (km)x$

x) **Existence of unity:** For any x, there exists $u = 1$ such that $u.x = 1.x = x^1 = x$.

Therefore, the set R^+ with the given operations forms a vector space.

3. If $V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x+y+z=1 \right\}$ with $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x+a-1 \\ y+b \\ z+c \end{pmatrix}$, $k \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} kx-k+1 \\ ky \\ kz \end{pmatrix}$

is a vector space. Then,

a) Find the zero element (additive identity)

b) The unity element (multiplicative identity)

c) The additive inverse of $v = \begin{pmatrix} -3 \\ 6 \\ -2 \end{pmatrix}$

d) the constants a, b, c if $\begin{pmatrix} a \\ 3 \\ -5 \end{pmatrix} \oplus \begin{pmatrix} b \\ -2 \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

Solution:

a) Let $\mathbf{O} = \begin{pmatrix} e \\ m \\ n \end{pmatrix}$ be the zero element . Then, for any vector $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we have

$$\mathbf{O} \oplus v = v \oplus \mathbf{O} = v \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} \oplus \begin{pmatrix} e \\ m \\ n \end{pmatrix} = \begin{pmatrix} x+e-1 \\ y+m \\ z+n \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\Rightarrow \begin{cases} x+e-1 = x \\ y+m = y \\ z+n = z \end{cases} \Rightarrow e = 1, m = 0, n = 0 \Rightarrow \mathbf{O} = \begin{pmatrix} e \\ m \\ n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

b) The unity element is the scalar k such that $k \otimes v = v$. But

$$k \otimes v = v \Rightarrow k \otimes \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} kx-k+1 \\ ky \\ kz \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{cases} kx-k+1 = x \\ ky = y \\ kz = z \end{cases} \Rightarrow k = 1$$

c) The additive inverse of v is an element u such that $u \oplus v = v \oplus u = \mathbf{0}$.

So, if we let $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, we get

$$u \oplus v = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \oplus \begin{pmatrix} -3 \\ 6 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a-3-1 \\ b+6 \\ c-2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 5 \\ -6 \\ 2 \end{pmatrix}$$

4. Consider the set $V = \left\{ \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} : a, b \in R \right\}$ with addition and scalar multiplication defined as follow:

$$\text{Addition}: \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \oplus \begin{pmatrix} c & 1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} a+c & 1 \\ 1 & b+d \end{pmatrix},$$

$$\text{Scalr multiplication}: k \otimes \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} = \begin{pmatrix} ka & 1 \\ 1 & kb \end{pmatrix}$$

a) Verify that V vector space

b) Find the zero element (additive identity)

c) The additive inverse of $M = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$

d) Compute $\left[2 \otimes \begin{pmatrix} 5 & 1 \\ 1 & -7 \end{pmatrix} \right] \oplus \begin{pmatrix} -6 & 1 \\ 1 & 7 \end{pmatrix}$

Solution:

a) Clearly, V is a vector space (Verify!)

b) Let $\mathbf{O} = \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix}$ be the zero element . Then, for any vector $W = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$,

$$\mathbf{O} \oplus M = M \oplus \mathbf{O} = M \Rightarrow \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \oplus \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+x & 1 \\ 1 & b+y \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} \Rightarrow \begin{cases} a+x=a \\ b+y=b \end{cases} \Rightarrow x=0, y=0$$

Hence, the zero element (additive identity) is $O = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

c) The additive inverse of M is an element W such that $M \oplus W = W \oplus M = \mathbf{O}$.

So, if we let $W = \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix}$, we get

$$M \oplus W = \begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix} \oplus \begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+3 & 1 \\ 1 & b-2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} a+3=0 \\ b-2=0 \end{cases} \Rightarrow a=-3, b=2 \Rightarrow W = \begin{pmatrix} -3 & 1 \\ 1 & 2 \end{pmatrix}$$

$$d) \left[2 \otimes \begin{pmatrix} 5 & 1 \\ 1 & -7 \end{pmatrix}\right] \oplus \begin{pmatrix} -6 & 1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 10 & 1 \\ 1 & -14 \end{pmatrix} \oplus \begin{pmatrix} -6 & 1 \\ 1 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & -7 \end{pmatrix}$$

3.2 Subspaces of a Vector space

Definition: A subset W of a vector space V over a field K is called a ***subspace*** of V if W itself is a vector space under the addition and scalar multiplication defined on V.

Test for a subspace: If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions are satisfied

(i) $\forall u, v \in W, u+v \in W$ (ii) If k is any scalar from K and **u** is any vector in W, then $ku \in W$. Conditions (i) and (ii) can be combined into a single criterion as follow:

If W is a sub set of the vector space V, then W is a subspace of V if and only if $\forall u, v \in W$ and for all scalars $a, b \in K$, $au + bv \in W$.

Examples:

1. Determine whether the following sets are subspaces or not for the vector space V with the usual operations.

a) $U = \{(x,0): x \in R\}$; $\mathbf{V} = R^2$ b) $W = \{(x,y,z): x-2y=0\}$; $\mathbf{V} = R^3$

c) $H = \{p \in P_3 : p(0) = 0\}$; $\mathbf{V} = P_3$ d) $M = \{A \in M_{3\times3}(\Re) : A = A^t\}$; $\mathbf{V} = M_{3x3}$

e) $F = \left\{ f : \int_a^b f(t)dt = 0 \right\}$; $\mathbf{V} = C[a,b]$(i.e. set of all continuous functions on [a,b])

Solutions:

a) Let $u,v \in U$ and k,m be any scalars. Then, it suffices to show $ku+mv \in U$.

But $u,v \in U \Rightarrow u=(x,0), v=(y,0)$. Thus,

$ku+mv=(kx,0)+(my,0)=(kx+my,0) \in U$. (Remember U consists of all vector in R^2 whose second coordinate is zero).

However, here, multiplying by scalar and adding any two elements of U does not alter the second coordinate from being zero. Therefore, U is a subspace of R^2.

b) Let $u,v \in W$ and k,m be any scalars. Then, it suffices to show $ku+mv \in W$.

But $u,v \in W \Rightarrow u=(x,y,z), x-2y=0, v=(a,b,c), a-2b=0$.

Thus, $ku+mv=(kx,ky,kz)+(ma,mb,mc)=(kx+ma,ky+mb,kz+mc)$.

(Remember that any vector in R^3 to be in W its first coordinate minus twice the second coordinate must be zero). In our case,

$$\begin{aligned}(kx+ma)-2(ky+mb) &= kx-2ky+ma-2mb \\ &= k(x-2y)+m(a-2b)=k.0+m.0=0\end{aligned}$$

Hence, $ku+mv=(kx+ma,ky+mb,kz+mc) \in W$. Therefore, W with the given condition is a subspace of R^3.

Note: In general, any subset of the form $W=\{(x,y,z): ax+by+cz=d\}$ is a subspace of $\mathbf{V}=R^3$ with the usual operations if and only if $d=0$ and it will not be a subspace if $d \neq 0$.

c) Let $f,p \in H$ and k,m be scalars. Then, it suffices to show $kf+mp \in H$. But $f,p \in H \Rightarrow f(0)=0, p(0)=0$.

Thus, $[kf+mp](0)=(kf)(0)+(mp)(0)=kf(0)+mp(0)$
$$=k.0+m.0=0\Rightarrow kf+mp\in H$$

Therefore, H is a subspace of $V=P_3$.

d) Let $A,B\in M$ and k,m be any scalars. Then, it suffices to show $kA+mB\in M$.

But $A,B\in M\Rightarrow A'=A, B'=B$. Thus, by properties of transpose, we have

$[kA+mB]'=(kA)'+(mB)'=kA'+mB'=kA+mB\Rightarrow[kA+mB]'=kA+mB$.

Hence, $kA+mB\in M$. Therefore, M is a subspace of $V=M_{3\times3}$.

e) Let $f,g\in F$ and k,m be any scalars. Then, it suffices to show $kf+mg\in F$.

But $f,g\in F\Rightarrow\int_a^b f(t)dt=0, \int_a^b g(t)dt=0$. Thus, by properties of definite integrals, we have

$$\int_a^b[kf+mg](t)dt=\int_a^b kf(t)dt+\int_a^b mg(t)dt=k\int_a^b f(t)dt+m\int_a^b g(t)dt=k.0+m.0=0.$$

This means for two continuous functions whose definite integral is zero on the given interval and so is their linear combination. Hence, $kf+mg\in F$.

Therefore, F is a subspace of $V=C[a,b]$.

3.3 Linear combinations and Spans

Definition: Let V be a vector space over a field K. Then, a vector $\mathbf{w}$ in V is said to be a linear combination of the vectors $v_1,v_2,\dots,v_n$ from V if and only if there exist scalars $c_1,c_2,\dots,c_n$ in K so that $\mathbf{w}=c_1v_1+c_2v_2+\dots+c_nv_n$.

Examples:

1.Determine whether $\mathbf{w}$ is a linear combination of v_1 and v_2.

a) $\mathbf{w}=(4,9)$, $v_1=(2,1)$, $v_2=(1,-3)$ in $V=R^2$

b) $\mathbf{w}=(1,5)$, $v_1=(1,3)$, $v_2=(2,6)$ in $V=R^2$

c) $\mathbf{w}=x^2+4$, $v_1=x^2+1$, $v_2=x^2-2$ in $V=P_2$

Solution:

a) $av_1 + bv_2 = \mathbf{w} \Rightarrow a(2,1) + b(1,-3) = (4,9)$

$$\Rightarrow 2a + b = 4,\ a - 3b = 9 \Rightarrow a = 3,\ b = -2$$

Therefore, $\mathbf{w} = 3v_1 - 2v_2 \Rightarrow (4,9) = 3(2,1) - 2(1,-3)$. This shows that $\mathbf{w}$ is a linear combination of v_1 and v_2.

b) $av_1 + bv_2 = \mathbf{w} \Rightarrow a(1,3) + b(2,6) = (1,5) \Rightarrow \begin{cases} a + 2b = 1 \\ 3a + 6b = 5 \end{cases}$

Subtracting three times the first from the second equation gives us $0 = 2$ which is a contradiction. This shows that $\mathbf{w}$ is not a linear combination of v_1 and v_2.

c) $av_1 + bv_2 = \mathbf{w} \Rightarrow x^2 + 4 = a(x^2 + 1) + b(x^2 - 2) \Rightarrow \begin{cases} a + b = 1 \\ a - 2b = 4 \end{cases} \Rightarrow a = 2,\ b = -1$

Therefore, $\mathbf{w} = 2v_1 - v_2 \Rightarrow x^2 + 4 = 2(x^2 + 1) - (x^2 - 2)$. Hence, $\mathbf{w}$ is a linear combination of v_1 and v_2.

2. Express the first vector as a linear combination of the two given vectors

a) $\mathbf{w} = (1,3,8),\ v_1 = (1,2,3),\ v_2 = (2,3,1)$

b) $f = 2x^2 - 3x + 5,\ g = 4x^2 + 1,\ h = 2x^2 + x - 1$

c) $M = \begin{pmatrix} -4 & -12 \\ 4 & 0 \end{pmatrix}; A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, C = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$

Solution: Notice that to find the linear combination of a vector say u in terms of other vectors say $v_1, v_2, \dots, v_n$ in a given vector space, simply find scalars $a_1, a_2, \dots, a_n$ from the field where the space is defined such that $u = a_1v_1 + a_2v_2 + \dots a_nv_n$.

a) $\mathbf{w} = av_1 + bv_2 \Rightarrow a(1,2,3) + b(2,3,1) = (1,3,8)$

$$\Rightarrow a + 2b = 1,\ 2a + 3b = 3, 3a + b = 8$$

Solving the system $\begin{cases} a + 2b = 1 \\ 2a + 3b = 3 \end{cases} \Rightarrow b = -1, a = 3$. Therefore, the linear combination expression is $\mathbf{w} = 3v_1 - v_2$.

b) $f = ag + bh \Rightarrow a(4x^2+1)+b(2x^2+x-1)$

$$= (4a+2b)x^2 + bx + a - b = 2x^2 - 3x + 5$$

Equating the coefficients of the same powers of x gives us the system

$$\begin{cases} 4a+2b=2 \\ b=-3 \\ a-b=5 \end{cases} \Rightarrow b=-3, a=2.$$

Therefore, the linear combination expression is $f = 2g - 3h$.

c) $M = aA + bB + cC + dD$

$$\Rightarrow a\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + c\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -4 & -12 \\ 4 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a+b+2c & -a+b+2c \\ b+3d & b+d \end{pmatrix} = \begin{pmatrix} -4 & -12 \\ 4 & 0 \end{pmatrix}$$

Using equality of matrices we get the systems $\begin{cases} b+3d=4 \\ b+d=0 \end{cases}$ and

$\begin{cases} a+b+2c=-4 \\ -a+b+2c=-12 \end{cases}$. Solving the first gives us $d=2, b=-2$ and using these values in the second system we get $\begin{cases} a+2c=-2 \\ -a+2c=-10 \end{cases}$ which give a gain $c=-3, a=4$. Therefore, $M = 4A - 2B - 3C + 2D$.

3. For what value of k, $u=(-1,k,-2)$ will be a linear combination of $v_1=(1,2,3)$ and $v_2=(2,3,1)$?

Solution: The vector $u=(-1,k,-2)$ will be the linear combination of $v_1=(1,2,3)$ and $v_2=(2,3,1)$ if there are scalars a and b such that

$u = av_1 + bv_2 \Rightarrow a(1,2,3)+b(2,3,1)=(-1,k,-2)$

$$\Rightarrow a+2b=-1,\ 2a+3b=k, 3a+b=-2$$

Solving the system $\begin{cases} a+2b=-1 \\ 3a+b=-2 \end{cases} \Rightarrow b=-1/5, a=-3/5$. Using these values in $2a+3b=k$, we get $k=-9/5$.

3.4 Span of a set of Vectors

Definition: Let $S = \{v_1, v_2, \ldots, v_n\}$ be the subset of a vector space V Then the span of S, usually denoted by $span(S)$, is the set of all possible linear combination of elements of S.

That is $span(S) = \{c_1v_1 + c_2v_2 + \ldots + c_nv_n : c_i \in R\}$.

Examples:

1. Describe the span of each of the following sets and interpret geometrically.

   a) $S = \{v_1 = (1,0,0),\ v_2 = (0,1,0)\}$ in $V = R^3$

   b) $S = \{v_1 = x^2 + 1,\ v_2 = x\}$ in $V = P_2$

Solution:

a) $span(S) = \{av_1 + bv_2 = a(1,0,0) + b(0,1,0) = (a,b,0) : a,b \in R\}$.

As we see here the span of S is all elements of R^3 with the z component being zero. Vectors like $(1,1,0)$, $(2,-3,0)$, $(0,0,0)$, $(0,1,0)$ and so on are in the span of S where as vectors like $(1,1,1)$, $(2,-3,-2)$, $(0,0,3)$ and so on are not in the span of S (because their third component is not zero). Geometrically, the span of S represents the $xy-$plane which is the subspace of R^3.

b) $span(S) = \{av_1 + bv_2 = a(x^2 + 1) + bx : a,b \in R\}$

2. Given the set of vectors $S = \{v_1 = (1,0,1), v_2 = (0,1,1)\}$

   i) Which of the following vectors belong to the span of S?

   a) $v = (2,3,5)$ b) $v = (4,-3,2)$ c) $v = (0,0,0)$ d) $v = (-3,3,0)$

   ii) For what value of t does $(8,6,5t-1)$ belongs to $span(S)$?

Solution: First let's find the span of the given set.

$$span(S) = \{av_1 + bv_2 : a,b \in R\}$$
$$= \{a(1,0,1) + b(0,1,1) : a,b \in R\} = \{(a,b,a+b) : a,b \in R\}$$

Here, observe that for a given vector $u = (x,y,z)$ to be in $span(S)$, its third component is the sum of the first and the second components, that is $u = (x,y,z) \in span(S) \Leftrightarrow z = x + y$.

In our case the vectors given in a, c, d belongs to $span(S)$ but the vector $v = (4,-3,2)$ (b) does not belong to $span(S)$ because the sum of the first and the second component is 1. Furthermore, the vector $(8,6,5t-1)$ will belong to $span(S)$ if $5t-1 = 8+6 \Rightarrow 5t = 15 \Rightarrow t = 3$.

3. For what value of k, does the vector $u = (1,k,13)$ belong to the span of the set $S = \{v_1 = (1,2,3), v_2 = (2,3,1)\}$?

Solution: The vector $u = (1,k,13)$ will be in the span of S if there are scalars a and b such that

$u = av_1 + bv_2 \Rightarrow a(1,2,3) + b(2,3,1) = (1,k,13) \Rightarrow a + 2b = 1,\ 2a + 3b = k, 3a + b = 13$

Solving the system $\begin{cases} a + 2b = 1 \\ 3a + b = 13 \end{cases} \Rightarrow b = -2, a = 5$.

Then, from $2a + 3b = k$, we get $k = 4$.

Remark: For any subset S of V, $span(S)$ is the subspace of V.

Definition: If $S = \{v_1, v_2, ..., v_n\}$ is the set of vectors in a vector space V, then the subspace W of V consisting of all linear combination of the vectors in S is said to be the space spanned by S and we say that the vectors $v_1, v_2, ..., v_n$ span W. It is written as $W = span(S)$ **or** $W = span\ (v_1, v_2, ..., v_n)$.

Here, S is called spanning or generating set.

Examples:

1. Find the subspace of R^3 spanned by $S = \{v_1 = (0,0,1), v_2 = (0,0,3), v_3 = (0,0,0)\}$

Solution: The subspace W spanned by S is given by

$W = span(S) = \{a(0,0,1) + b(0,0,3) + c(0,0,0)\} = \{(0,0,a+3b) : a, b \in R\}$

2. Show that R^2 is spanned by $S = \{v_1 = (1,2), v_2 = (3,0)\}$.

Solution: It suffices to show that for arbitrary vector $v = (x,y)$ in R^2, there exists scalars a and b (they could be expressed in terms of x and y) such that $v = (x,y)$ is a linear combination of them.

Here,

$$v = av_1 + bv_2 \Rightarrow a(1,2) + b(3,0) = (x,y) \Rightarrow \begin{cases} a + 3b = x \\ 2a = y \end{cases} \Rightarrow a = \frac{y}{2},\ b = \frac{2x-y}{6}.$$

Hence, for arbitrary vector $v = (x,y)$ in R^2, there exists scalars $a = \frac{y}{2},\ b = \frac{2x-y}{6}$ (in terms of the components of vector v) such that $v = (x,y) = \frac{y}{2}(1,2) + \frac{2x-y}{6}(3,0)$. For instance for $v = (1,8)$, there are scalars $a = \frac{y}{2} = 4,\ b = \frac{2x-y}{6} = -1$ such that $v = (1,8) = 4(1,2) - (3,0)$.

Therefore, the set $S = \{v_1 = (1,2), v_2 = (3,0)\}$ spans R^2 or R^2 is spanned by $S = \{v_1 = (1,2), v_2 = (3,0)\}$. That is $R^2 = span\{(1,2),(3,0)\}$.

3. Suppose $S = \{(1,0,k),\ (1,2,-3),(k,1,0)\}$. Then, for what value of k does $span(S) = R^3$?

Solution: Given arbitrary vector $w = (x,y,z)$ in R^3, we need to find unique constants $a,\ b,\ c$ such that $a(1,0,k) + b(1,2,-3) + c(k,1,0)\} = (x,y,z)$.

After rearrangement, we get the system $\begin{cases} a + b + ck = x \\ 2b + c = y \\ ak - 3b = z \end{cases}$

In matrix form, this is equivalent to the system $\begin{pmatrix} 1 & 1 & k \\ 0 & 2 & 1 \\ k & -3 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Now, this system will have a unique solution for $a,\ b,\ c$ if the coefficient matrix

$M = \begin{pmatrix} 1 & 1 & k \\ 0 & 2 & 1 \\ k & -3 & 0 \end{pmatrix}$ is non-singular. i.e $\det M = \begin{vmatrix} 1 & 1 & k \\ 0 & 2 & 1 \\ k & -3 & 0 \end{vmatrix} \neq 0 \Rightarrow k \neq 3/2, -1$.

Therefore, $span(S) = R^3$ if $k \neq 3/2, -1$.

3.5 Linearly Independent and Dependent Vectors

Definition: Let $S=\{v_1,v_2,\dots,v_n\}$ be the subset of a vector space V. Then, S is said to be linearly independent set (the vectors in S are linearly independent) if $c_1v_1+c_2v_2+\dots+c_nv_n=\mathbf{0}\Rightarrow c_i=0$ for all **i**, then S is linearly independent set. If $c_1v_1+c_2v_2+\dots+c_nv_n=\mathbf{0}\Rightarrow c_i\neq 0$ for some **i**, then S is linearly dependent set.

Examples:

1. Check whether the following sets of vectors are linearly independent or dependent

a) $S=\{(1,3,0),(-1,1,1)\}$ *b)* $S=\{(2,-4),(-3,6)\}$ *c)* $S=\{2,\ 1+x,\ x^2\}$

Solution:

a) $c_1(1,3,0)+c_2(-1,1,1)=\mathbf{0}\Rightarrow(c_1-c_2,3c_1+c_2,c_2)=(0,0,0)$

$$\Rightarrow\begin{cases}c_1-c_2=0\\3c_1+c_2=0\Rightarrow c_1=c_2=0\\c_2=0\end{cases}$$

Hence, the set S is linearly independent.

b) $c_1(2,-4)+c_2(-3,6)=\mathbf{0}\Rightarrow(2c_1-3c_2,6c_2-4c_1)=(0,0)$

$$\Rightarrow\begin{cases}2c_1-3c_2=0\\6c_2-4c_1=0\end{cases}\Rightarrow c_2=\frac{2}{3}c_1$$

For instance if we take, $c_1=6$, we get $c_2=\frac{2}{3}c_1=4$. Then for these values $c_1(2,-4)+c_2(-3,6)=6(2,-4)+4(-3,6)=(0,0)$. This means we have got non-zero scalars such that the linear combination of elements of S is zero. Hence, S is linearly dependent.

c) $$2c_1+c_2(1+x)+c_3x^2=0\Rightarrow\begin{cases}2c_1+c_2=0\\c_2=0\\c_3=0\end{cases}\Rightarrow c_1=c_2=c_3=0.$$

Hence, the set S is linearly independent.

2. For which λ are the vectors $v_1 = (\lambda,-1,-1), v_2 = (-1,\lambda,-1), v_3 = (-1,-1,\lambda)$ linearly dependent?

Solution: The vectors v_1, v_2, v_3 will be linearly dependent if and only if there exists scalars a,b,c not all zero such that $av_1 + bv_2 + cv_3 = 0$. That is

$$\begin{cases} a\lambda - b - c = 0 \\ -a + \lambda b - c = 0 \\ -a - b + \lambda c = 0 \end{cases}$$

must have more than one solution. But this system will have many solution if and only if the cooefficent matrix is singular. That is its determinant is zero. Thus,

$$\begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = 0 \Rightarrow (\lambda+1)^2(\lambda-2) = 0 \Rightarrow \lambda = -1,2.$$

3.6 Basis and Dimensions of Vector Spaces

Definition: The set $B = \{v_1, v_2, ..., v_n\}$ is a basis for a vector space V if and only if the following two conditions are satisfied:

i) The set B is linearly independent ii) $span(B) = V$.

The number of elements in a basis B of a vector space V is called **dimension of** V and denoted by $\dim V$. If the dimension of a vector space is finite, then we say that it is finite dimensional. That is if the dimension of a vector space is n, then we say that it is n-dimensional space. Otherwise, it is infinite dimensional.

Examples:

1. Check whether the following sets are basis for the given vector space V

a) $B = \{(2,4),(1,1)\}, V = R^2$ b) $B = \{(1,0,1),(0,-1,1),(0,2,0)\}, V = R^3$

c) $B = \{1, x, x + x^2\}, V = P_2$

Solution: In each case, we must check the two conditions in the definition.

a) Checking *linearly independence:*

$$a(2,4) + b(1,1) = (0,0) \Rightarrow \begin{cases} 2a + b = 0 \\ 4a + b = 0 \end{cases} \Rightarrow a = b = 0.$$

Then , the two vectors are linearly independent. Besides, for any vector $v=(x,y)$ in $R^2, a(2,4)+b(1,1)=(x,y) \Rightarrow \begin{cases} 2a+b=x \\ 4a+b=y \end{cases} \Rightarrow a=\frac{y-x}{2}, b=2x-y$.

This means for any vector $v=(x,y)$ in R^2, we can find scalars

$a=\frac{y-x}{2}, b=2x-y$ such that $v=(x,y)=a(2,4)+b(1,1)$.

Therefore, the given vectors are basis of $V=R^2$ and $\dim V=2$.

b) Checking *linearly independence:*

$$a(1,0,1)+b(0,-1,1)+c(0,2,0)=(0,0,0) \Rightarrow \begin{cases} a=0 \\ 2c-b=0 \\ a+b=0 \end{cases} \Rightarrow a=b=c=0.$$

Then , the given vectors are linearly independent.

Besides, for any vector $v=(x,y,z)$ in R^3,

$a(1,0,1)+b(0,-1,1)+c(0,2,0)=(x,y,z)$

$$\Rightarrow \begin{cases} a=x \\ 2c-b=y \\ a+b=z \end{cases} \Rightarrow a=x, b=z-x, c=\frac{y+z-x}{2}$$

This means for any vector $v=(x,y,z)$ in R^3, we can find scalars

$a=x, b=z-x, c=\frac{y+z-x}{2}$ such that $v=(x,y,z)=a(1,0,1)+b(0,-1,1)+c(0,2,0)$

Therefore, the given vectors are basis of the vector space $V=R^3$ and $\dim V=3$.

c) Checking *linearly independence:*

$a+(b+c)x+cx^2=0 \Rightarrow \begin{cases} a=0 \\ b+c=0 \\ c=0 \end{cases} \Rightarrow a=b=c=0$. Then , the given vectors are linearly independent.

Besides, for any polynomial vector $p=a_0+a_1x+a_2x^2$ in P_2,

$a+bx+c(x+x^2)=a_0+a_1x+a_2x^2 \Rightarrow a=a_0, b=a_1-a_2, c=a_2$.

This means that every polynomial is expressible as a linear combination of the given vectors. Therefore, the given vectors are basis of the vector space $V = P_2$ and $\dim V = 3$.

2. Find a basis and the dimension of the subspace $W = \{(x, y, z, 0,0,0) : x - 6y = 0, x = 3z : x, y, z \in R\}$

Solution: Here,

$$x - 6y = 0, x = 3z \Rightarrow x = 6y, z = \frac{x}{3} = 2y \Rightarrow (6y, y, 2y, 0,0,0) = (6,1,2,0,0,0)y$$

Hence, $W = \{(x, y, z, 0,0,0) : x - 6y = 0, x = 3z\} = \{(6,1,2,0,0,0)y : y \in R\}$. Thus the basis of W becomes $B = \{(6,1,2,0,0,0)\}$ and its dimension is $\dim W = 1$.

3. Find a basis and the dimension of the following subspaces.

a) $W = \{(x, y, z) : x - y + z = 0 : x, y, z \in R\}$

b) $W = \left\{ A \in M_{2\times 2}(R) : A = \begin{pmatrix} a & 2c \\ 2c & -b \end{pmatrix} \right\}$

c) $W = \{p(x) : \frac{d}{dx}(p(x)) = 0\}$ d) $W = \{p \in P_2 : p(1) = p(-1) = 0\}$

Solution:

a) Here, $x - y + z = 0 \Rightarrow z = y - x$ such that

$(x, y, z) = (x, y, y - x) = (x, 0, -x) + (0, y, y) = (1,0,-1)x + (0,1,1)y$

Now, verify that the set $B = \{(1,0,-1), (0,1,1)\}$ is linearly independent.

So, the basis of W is $B = \{(1,0,-1), (0,1,1)\}$ and $\dim W = 2$.

b) Here, $A = \begin{pmatrix} a & 2c \\ 2c & -b \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$

Thus, $W = \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} + c\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} : a, b, c \in R \right\}$.

Now, verify that the set $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \right\}$ is linearly independent.

Thus, the basis of W is $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \right\}$ and $\dim W = 3$.

c) $W = \{p(x) : \frac{d}{dx}(p(x)) = 0\} = \{a + bx + cx^2 + dx^3 : b + 2cx + 3dx^2 = 0\}$

$= \{a + bx + cx^2 + dx^3 : b = c = d = 0\} = \{a : a \in R\}$

Hence, the basis of W becomes $B = \{1\}$ and $\dim W = 1$

d) Let $p(x) = a_0 + a_1x + a_2x^2$ in P_2 be arbitrary.

Then, $p(1) = p(-1) = 0 \Rightarrow \begin{cases} a_0 + a_1 + a_2 = 0 \\ a_0 - a_1 + a_2 = 0 \end{cases} \Rightarrow a_2 = -a_0, a_1 = 0$

Hence, $p(x) = a_0 + a_1x + a_2x^2 = a_0(1 - x^2)$.

Therefore, $W = \{p(x) = a_0(1 - x^2) : a_0 \in R\}$

Thus, the basis of W is $B = \{1 - x^2\}$ and $\dim W = 1$.

4. Consider the subset W of the vector space $V = M_{2\times2}$ given by $W = \{A \in M_{2\times2} : A = A'\}$. Show that W is the subspace of V and give the basis and dimension of W.

Solution: Since the zero vector is symmetric, it belongs to W. Thus, W is non-empty. Now let A and B be any two matrices in W. We need to show their sum $A + B$ is also in W. Here, $A, B \in W \Rightarrow A = A', B = B'$. Besides from the property of transpose, we have that $(A + B)' = A' + B' \Rightarrow (A + B)' = A + B \Rightarrow A + B \in W$

On the other hand, for any scalar k, $(kA)' = kA' = kA \Rightarrow kA \in W$. Thus, by the test of subspace, the subset $W = \{A \in M_{2\times2} : A = A'\}$ which is the set of all 2×2 symmetric matrices is the subspace of the vector space of all 2×2 matrices. Finally, let's determine the basis and dimension of W. Since W is the set of symmetric matrices, any matrix A in W is of the form

$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. But, $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

Thus, $W = \left\{ a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} : a, b, c \in R \right\}$.

Therefore, the basis is $B = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$

3.7 Relative Coordinates (Coordinate Vectors)

Definition: Suppose $B=\{v_1,v_2,...,v_n\}$ is basis of the vector space V. If $v=c_1v_1+c_2v_2+...+c_nv_n$ is the expression of the vector v in V in terms of the basis, then the scalars $c_1,c_2,...,c_n$ are called coordinates of v relative to the basis B. The vector $w=(c_1,c_2,...,c_n)$ formed by these scalars is called coordinate vector of v relative to B and it is denoted by $v_B=(c_1,c_2,...,c_n)$.

Examples:

1. Find the coordinates of the given vector v relative to the given basis B and give the coordinate vectors (Assume the usual vector spaces in each case).

a) $v=(5,-6,-1);\ B=\{(1,0,1),(0,1,1)\}$

b) $v=2-x,\ B=\{1-x,1+x\}$

c) $v=\begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}, B=\left\{\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix},\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}\right\}$

d) $v=(2,0,-2); B=\{(-1,0,1),(3,0,5)\}$

Solution:

a) Here, $(5,-6,-1)=a(1,0,1)+b(0,1,1)\Rightarrow a=5, b=-6$. Hence, $v_B=(5,-6)$

b) Here,

$$2-x=a(1-x)+b(1+x)\Rightarrow a+b=2, b-a=-1\Rightarrow a=\frac{3}{2}, b=\frac{1}{2}\Rightarrow v_B=(\frac{3}{2},\frac{1}{2}).$$

c) Here, $\begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}=a\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}+b\begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}+c\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}+d\begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}$

$$\Rightarrow\begin{cases}a+c=-1\\ a+d=1\\ b=2\\ b+d=1\end{cases}\Rightarrow v_B=(2,2,-3,-1)$$

d) Since the vectors are not basis of R^3, we cannot find the coordinates of v.

2. Find the coordinates of

a) $f = x^2 + x + 1$ with respect to $B = \{1, x-1, (x-2)(x-1)\}$

b) $f = \cos(x + \frac{\pi}{6})$ with respect to $B = \{g = 2\sin x, h = \sqrt{3}\cos x\}$

c) $f = \ln\left(\frac{9x^4 + 6x^2 + 1}{e^{3x}}\right)$ with respect to $B = \{g = \ln(3x^2+1),\ h = \sqrt{2}x\}$

Solution:

a) We need to find scalars a, b, c such that

$a + b(x-1) + c(x-2)(x-1) = f$. But

$$a + b(x-1) + c(x-2)(x-1) = f \Rightarrow a + b(x-1) + c(x^2 - 3x + 2) = x^2 + x + 1$$

$$\Rightarrow a - b + 2c + (b - 3c)x + cx^2 = x^2 + x + 1$$

$$\Rightarrow \begin{cases} a - b + 2c = 1 \\ b - 3c = 1 \\ c = 1 \end{cases} \Rightarrow b = 4, a = 3, c = 1$$

$$\Rightarrow (f)_B = (3,4,1)$$

b) We need to find scalars a, b such that $ag + bh = f$.

But note that $f = \cos(x + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$.

Thus, $ag + bh = f \Rightarrow 2a\sin x + \sqrt{3}b\cos x = \cos(x + \frac{\pi}{6}) = \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x$

$$\Rightarrow 2a = -\frac{1}{2}, \sqrt{3}b = \frac{\sqrt{3}}{2} \Rightarrow a = -\frac{1}{4}, b = \frac{1}{2} \Rightarrow (f)_B = (-\frac{1}{4}, \frac{1}{2})$$

c) We need to find scalars a, b such that $ag + bh = f$. But note that

$$f = \ln\left(\frac{9x^4 + 6x^2 + 1}{e^{3x}}\right) = \ln\left(\frac{(3x^2+1)^2}{e^{3x}}\right) = \ln(3x^2+1)^2 - \ln e^{3x} = 2\ln(3x^2+1) - 3x$$

Thus, $ag + bh = f \Rightarrow a\ln(3x^2+1) + \sqrt{2}bx = 2\ln(3x^2+1) - 3x$

$$\Rightarrow a = 2, \sqrt{2}b = -3 \Rightarrow a = 2, b = -\frac{3}{\sqrt{2}} \Rightarrow (f)_B = (2, -\frac{3}{\sqrt{2}})$$

3. Which of the following is not in the span of $S=\{\cos^2 x,\sin^2 x\}$? For these which are in the span, find the coordinate vectors.

a) $f(x)=5$ *b)* $f(x)=3\sin^2 x$ *c)* $f(x)=\cos(2x)$ *d)* $f(x)=\sin 2x$

Solution: Any function f is in the span of $S=\{\cos^2 x,\sin^2 x\}$ if and only if $f=a\cos^2 x+b\sin^2 x$ for some constants a and b.

a) $f(x)=5=a\cos^2 x+b\sin^2 x$ for $a=b=5$. So, $f(x)=5$ is in the span of S and the coordinate vector is $(5,5)$.

b) $f(x)=3\sin^2 x=a\cos^2 x+b\sin^2 x$ for $a=0,b=3$. That means $f(x)=3\sin^2 x$ is in the span of S and the coordinate vector is $(0,3)$.

c) $f(x)=\cos(2x)=\cos^2 x-\sin^2 x=a\cos^2 x+b\sin^2 x$ for $a=1,b=-1$. That means $f(x)=\cos(2x)$ is in the span of S and the coordinate is $(1,-1)$.

d) $f(x)=\sin 2x=2\sin x\cos x$ can never be expressed in the form $\sin(2x)=a\cos^2 x+b\sin^2 x$. That means it is not in the span of S.

Remarks: The coordinate vector of a given vector depends both on the basis given and the order in which the basis element are placed. That means the change in the order of the basis vectors results in a change in the order of the entries of the coordinates vectors.To find the vector whose coordinate vector is given, just write out the linear combination of the vectors using the entries of the coordinate vector as coefficients in the respective order. Remember that the first entry is the coefficient of the first vector listed in the given set, the second is the coefficient for the second vector listed and so on.

Example: Find the vector v if

a) $v_B=(3,2,-3),\quad B=\{v_1=(1,2,0),v_2=(3,-5,2),v_3=(0,1,3)\}$

b) $v_{B'}=(3,2,-3),\quad B'=\{v_3=(0,1,3),v_2=(3,-5,2),v_1=(1,2,0)\}$

Solution: Here, notice that the same coordinate vector is given but the order of the sets of vector is changed. So, let's see what happens to the value of v.

a) $v=3v_1+2v_2-3v_3=3(1,2,0)+2(3,-5,2)-3(0,1,3)=(9,-7,-5)$

b) $v=3v_3+2v_2-3v_1=3(0,1,3)+2(3,-5,2)-3(1,2,0)=(3,-13,13)$

Review Problems on Chapter-3

1. Suppose $V=\{(x,y):x,y\in R\}$. Define addition and scalar multiplication as follow. Verify that this is a vector space over R. Give the zero element.
$(x,y)\oplus(a,b)=(x+a+1,y+b+1)$, $k\otimes(x,y)=(kx+k-1,ky+k-1)$.

Answer : The zero element is $O=(-1,-1)$

2*. Consider a vector space V as given below.

$$V=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}:x+y+z=5\right\}\text{with}\begin{pmatrix}x\\y\\z\end{pmatrix}\oplus\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}x+a\\y+b-5\\z+c\end{pmatrix},k\otimes\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}kx\\ky-5k+5\\kz\end{pmatrix}$$

Then, Find

a) The zero (additive identity) b) The unity (multiplicative identity)

c) The additive inverse of $v=\begin{pmatrix}6\\-8\\7\end{pmatrix}$ c) compute $\begin{pmatrix}-3\\1\\7\end{pmatrix}\oplus 3\otimes\begin{pmatrix}-3\\2\\6\end{pmatrix}$.

Answer: $a)\begin{pmatrix}0\\5\\0\end{pmatrix}$ $b)\ u=1$ $c)\begin{pmatrix}-6\\18\\-7\end{pmatrix}$ $d)\begin{pmatrix}-12\\-8\\25\end{pmatrix}$

3. Check whether the sets are subspaces of the indicated vector space or not.

a) $W=\{(x,y,z):x+y+z=0:x,y,z\in R\};V=R^3$

b) $W=\{(x,y,0):x,y\in R\};V=R^3$ c) $W=\{p\in P_2:p(1)=p(-1)\}$

d) $W=\{p\in P_3:p'(1)=0\}$ e) $W=\{(x,y,z):x+y+z\le 1\};V=R^3$

g) $W=\{(x,y,z):z\le 0\};V=R^3$ h) $W=\{(x,y,1)\};V=R^3$

5. Find t such that $v=(6,5t-1,16)$ belongs to the span of
$S=\{v_1=(2,3,0),v_2=(0,-5,4)\}$ **Answer:** $t=-2$

6. Find k such that $span\{(1,-1,k),(1,0,1),(2,1,-1)\}\ne R^3$. **Answer:** $k=4$

7. For what value of k, does the vector $u=(1,k,13)$ belong to the span of
$S=\{v_1=(1,2,3),v_2=(2,3,1)\}$?

8. If the set $W = \{(x, y, z) : x - 2y + 3z = 2a - 5\}$ is the subspace of the vector space $\mathbf{V} = R^3$ with the usual operations, what must be the value of a?

Answer: $a = 5/2$.

9. For what values of h are the vectors $u = (-1,-4,3), v = (4,5,0), w = (10,7,h+3)$ are linearly dependent in R^3? **Answer** : $h = 3$

10. Find a basis and the dimension of the following subspaces.

a) $W = span\{(0,0,0),(1,-1,0),(1,2,1),(-3,3,0)\}$

b) $W = \{(x, y, z) : x + 2y - z = 0 : x, y, z \in R\}$

11*. Find the coordinates of

a) $f = x^2 + x + 1$ with respect to $B = \{1, x-1, (x-2)(x-1)\}$

b) $f = \ln\left(\dfrac{9x^4 + 6x^2 + 1}{e^{3x}}\right)$ with respect to $B = \{g = \ln(3x^2 + 1),\ h = \sqrt{2}x\}$.

c) $f(x) = \cos(x + \dfrac{\pi}{6})$ with respect to $g = 2\sin x$ and $h = \sqrt{3}\cos x$

12. For what values of r are $v_1 = (r,1,1), v_2 = (1,r,1), v_3 = (1,1,r)$ a basis for R^3?

Answer : $r \neq 1, -2$

13. Find the bases and dimensions for the subspace of P_2 given by

a) $U = \{p(x) : p(0) = 0\}$ b) $W = \{p(x) : \dfrac{d}{dx}(p(0)) = 0\}$

Answer : a) $B = \{x, x^2\}, \dim U = 2$ b) $B = \{1, x^2\}, \dim W = 2$

14*. Find the bases and dimensions for the subspace W of P_3 given by

$W = \{p(t) : \int_{-1}^{1} p(t)dt = 0\}$ **Answer** : $B = \{x^3, 1 - 3x^2, x\}$; $\dim W = 3$

15. Suppose $W = \{p \in P_4 : p(0) = 0$ and $p'(0) = 0\}$ is a subspace of P_4. Find the basis and dimension of W. **Answer** : $B = \{x^4, x^3, x^2\}$, $\dim W = 3$

16*. Find the basis for the subspace of P_4 spanned by $f \in P_4$ that satisfies $f(x) = x^4 f(\frac{1}{x})$ **Answer** : $B = \{x^4 + 1, x^3 + x, x^2\}$.

CHAPTER-4

Revisions on Limit and Continuity

4.1 Definition and Examples

Here, under we are going to revise the concept of limit and limit computations. Of course, we know that without limit calculus would not exist, every idea of differential or integral calculus is the generalization of the limiting process in one way or another. For instance, the slope of a curve is nothing but the limit of the secant line, the area of a region is nothing but the limit of the sum of the approximating rectangles, many, many more results of calculus involve limiting process.

Definition: Let L be a real number. Then we say that L is the limit of $f(x)$ at a if and only if $f(x)$ gets closer and closer to L as x approaches to a ($f(x)$ may not be defined at a and x also may not be equal to a). This is denoted by $\lim_{x\to a} f(x) = L$.

One Sided Limits: Note that when we say x approaches to a, we mean either from above or from below. For instance for $a = 2$, if we take $x = 1.9$, $x = 1.99$, $x = 1.999$ and so on in this case x approaches to $a = 2$ from the left (from below). On the other hand, if we take $x = 2.1$, $x = 2.01$, $x = 2.001$ and so on in this case, x approaches to $a = 2$ from the right (from above). Here, the limit of the function $f(x)$ as x approaches to a from the left is called left hand limit and denoted by $\lim_{x\to a^-} f(x) = L$. (In this case, we used the notation $x \to a^-$ in the sense that x is slightly less than that of a or to mean $x < a$). Again, the limit of the function $f(x)$ as x approaches to a from the right is called right hand limit and denoted by $\lim_{x\to a^+} f(x) = R$. (In this case, we used the notation $x \to a^+$ in the sense that x is slightly greater than that of a or to mean $x > a$).

Always, think the limit of a function at a point is just the approximate value of the function near to that point. For example, if take $f(x) = x - 2$ and $a = 2$, then the value of this function at $x = 1.9$, $x = 1.99$, $x = 1.999$ is approaches to zero and at $x = 2.1$, $x = 2.01, x = 2.001$ is also approaches zero. But not exactly zero at these points. Thus, $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = 0$.

Notes: If $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$, then we say that the limit of $f(x)$ as x approaches to a exists and denoted by $\lim_{x\to a} f(x) = L$. If $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then we say that the limit of $f(x)$ as x approaches to a does not exist.

Examples:

1. Evaluate the following limits (if it exists)

$a)$ $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} x+1, & x > 1 \\ x-1, & x \le 1 \end{cases}$ $\quad b)$ $\lim_{x\to 3} f(x)$ where $f(x) = |x-3|$

Solution: a) Here, $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(x+1) = 2$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-}(x-1) = 0$. But, $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$ which means that $\lim_{x\to 1} f(x)$ does not exist.

b) Here, $f(x) = |x-3| = \begin{cases} x-3, x \ge 3 \\ 3-x, x < 3 \end{cases}$.

So, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}(3-x) = 0$, $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+}(x-3) = 0$

$$\Rightarrow \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 0 \Rightarrow \lim_{x\to 3} f(x) = 0$$

2. In each case, evaluate $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$. Does $\lim_{x\to 0} f(x)$ exist?

$a)$ $f(x) = \dfrac{3x + |x|}{7x - 5|x|}$ $\qquad b)$ $f(x) = \dfrac{x - |x|}{x}$

Solution: First, re-write the functions using the properties of absolute value

$a)$ $f(x) = \begin{cases} 2; x > 0 \\ \frac{1}{6}; x < 0 \end{cases} \Rightarrow \lim_{x\to 0^-} f(x) = \frac{1}{6}, \lim_{x\to 0^+} f(x) = 2 \Rightarrow \lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

Hence, $\lim_{x\to 0} f(x)$ does not exist.

Basic Limit Theorems: Usually we face limit problems involving combination of different forms of functions. In that case, we need a way how to evaluate the limit of such combinations of functions. Mostly, we apply the following limit theorems to evaluate limits of combinations. Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exists say, $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$ and let k be any real number. Then,

a) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L + M$

b) $\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = LM$

c) $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kL$.

Examples:

1. If $\lim_{x\to a} f(x) = 3, \lim_{x\to a} g(x) = -2$, evaluate $\lim_{x\to a} \frac{3f(x) - 4g(x)}{5f(x) + 2g(x)}$.

Solution:

$$\lim_{x\to a} \frac{3f(x) - 4g(x)}{5f(x) + 2g(x)} = \frac{\lim_{x\to a}[3f(x) - 4g(x)]}{\lim_{x\to a}[5f(x) + 2g(x)]} = \frac{3\lim_{x\to a} f(x) - 4\lim_{x\to a} g(x)}{5\lim_{x\to a} f(x) + 2\lim_{x\to a} g(x)} = \frac{9+8}{15-6} = \frac{17}{9}.$$

2*. If $\lim_{x\to 6} \frac{f(x)-8}{x-6} = 10$, then find $\lim_{x\to 6} f(x)$.

Solution:

$$\lim_{x\to 6} \frac{f(x)-8}{x-6} = 10 \Rightarrow \frac{\lim_{x\to 6}[f(x)-8]}{\lim_{x\to 6}(x-6)} = 10 \Rightarrow \lim_{x\to 6}[f(x)-8] = \lim_{x\to 6} 10(x-6) = 0$$

$$\Rightarrow \lim_{x\to 6} f(x) - 8 = 0 \Rightarrow \lim_{x\to 6} f(x) = 8$$

4.2 Limit at Infinity

Definition: We say that a real number L is the limit of a function f as x approaches to infinity if and only if $f(x)$ gets closer to L as x increases without bound. This is denoted by $\lim_{x\to\infty} f(x) = L$.

Computation of limits at infinity: Suppose we want to evaluate $\lim_{x\to\infty}\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials or n^{th} roots of polynomials. Then, to evaluate $\lim_{x\to\infty}\frac{p(x)}{q(x)}$ first divide $p(x)$ and $q(x)$ by the highest power of x. The highest power of x may be found in $p(x)$ or $q(x)$. Note that if $f(x)$ is a polynomial of degree n, the highest power of x in $\sqrt[m]{f(x)}$ is $\frac{n}{m}$.

For instance, in $\sqrt{f(x)}$, the highest power of x is $\frac{n}{2}$ in $\sqrt[3]{f(x)}$, the highest power of x is $\frac{n}{3}$. Besides, we use the basic limits $\lim_{x\to\infty}\frac{1}{x} = 0$, $\lim_{x\to\infty} x = \infty$ and it is also useful to remember as $x \to +\infty$, since $x > 0$, $x = \sqrt{x^2}$ and

as $x \to -\infty$, since $x < 0$, $x = -\sqrt{x^2}$

Examples: Evaluate the following limits using the above procedures.

a) $\lim_{x\to\infty}\frac{2x^4+4}{x^4+1}$ b) $\lim_{x\to\infty}\frac{x^2+1}{x^3+3x^2+2}$ c) $\lim_{x\to\infty}\frac{\sqrt{3x^4+1}}{x+3}$ d) $\lim_{x\to\infty}\frac{\sqrt{x^2+3}}{2x+5}$

Solution:

a) Here, the highest power in the numerator and denominator is x^4.

So, $\lim_{x\to\infty}\frac{2x^4+4}{x^4+1} = \lim_{x\to\infty}\frac{2+4/x^4}{1+1/x^4} = 2$.

b) Here, the highest power is x^3 (found in the denominator).

So, $\lim_{x\to\infty}\frac{x^2+1}{x^3+3x^2+2}=\lim_{x\to\infty}\frac{1/x+1/x^3}{1+3/x+2/x^3}=0$.

c) The higehst power of x is $\frac{4}{2}=2$.

So, $\lim_{x\to\infty}\frac{\sqrt{3x^4+1}}{x+3}=\lim_{x\to\infty}\frac{(\sqrt{3x^4+1})/x^2}{(x+3)/x^2}=\lim_{x\to\infty}\frac{\sqrt{3+1/x^4}}{1/x+3/x^2}=0$

d) The higehst power of x is $\frac{2}{2}=1$.

So $\lim_{x\to\infty}\frac{\sqrt{x^2+3}}{2x+5}=\lim_{x\to\infty}\frac{(\sqrt{x^2+3})/x}{(2x+5)/x}=\lim_{x\to\infty}\frac{(\sqrt{x^2+3/x^2})}{(2+5/x)}=\frac{1}{2}$

4.3 Evaluation of Limits (Techniques of Computation of Limits)

To evaluate the limits of different functions or combinations of functions, we use different techniques. In what follows I will state the most commonly used techniques depending on the situation and explain how to use each techniques using examples.

i) By Direct Substitution: This is a method used to evaluate limit $\lim_{x\to a} f(x)$ by substituting $x=a$ directly in the formula of f .That is $\lim_{x\to a} f(x)=f(a)$ .This method usually works when the function is defined at $x=a$.

Examples: Evaluate the limits $a)$ $\lim_{x\to 2}(x^2+1)$ $\quad b)$ $\lim_{x\to 3^+}\frac{x-2}{x+5}$

Solution:

$a)\lim_{x\to 2}(x^2+1)=(2^2+1)=5$ $\quad b)$ $\lim_{x\to 3^+}\frac{x-2}{x+5}=\frac{3-2}{3+5}=\frac{1}{8}$

ii) Factorization followed by cancellation: This is a method of evaluating limits which works as follow. First factorize the function. Second cancel a term that *creates* a problem to evaluate the limits. Finally apply direct substitution. To use this method, we must recall the following basic factorization rules.

$i)$ $x^2-y^2=(x-y)(x+y)$

$ii)$ $x-y=(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y}), x,y\geq 0$

$iii)$ $x^3-y^3=(x-y)(x^2+xy+y^2)$

$iv)$ $x^3+y^3=(x+y)(x^2-xy+y^2)$

$v)$ $x^4-y^4=(x-y)(x+y)(x^2+y^2)$

Examples:

1. Evaluate the following limit

$a)\lim_{x\to 2}\frac{x^2-4}{x-2}$ $\quad b)\lim_{x\to 3}\frac{x^2-3x}{\sqrt{x}-\sqrt{3}}$ $\quad c)\lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)$ $\quad d)\lim_{x\to \frac{-1}{2}}\frac{8x^3+1}{2x+1}$

$e)\lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x^3-27}$ $\quad f)\lim_{x\to 2}\frac{x^3-8}{x^2-4}$ $\quad g)\lim_{x\to 4}\frac{2x-8\sqrt{x}+8}{2-\sqrt{x}}$ $\quad h)\lim_{x\to 1}\frac{1-1/x}{1-x^2}$

Solution : $a)\ \lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{x-2}=\lim_{x\to 2}(x+2)=4$

$b)\ \lim_{x\to 3}\frac{x^2-3x}{\sqrt{x}-\sqrt{3}}=\lim_{x\to 3}\frac{x(x-3)}{\sqrt{x}-\sqrt{3}}=\lim_{x\to 3}\frac{x(\sqrt{x}-\sqrt{3})(\sqrt{x}+\sqrt{3})}{\sqrt{x}-\sqrt{3}}=6\sqrt{3}$

$c)\ \lim_{x\to 2}\left(\frac{1}{x-2}-\frac{4}{x^2-4}\right)=\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right)=\lim_{x\to 2}\frac{(x-2)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{1}{x+2}=\frac{1}{4}$

$d)\ \lim_{x\to\frac{-1}{2}}\frac{8x^3+1}{2x+1}=\lim_{x\to\frac{-1}{2}}\frac{(2x)^3+1}{2x+1}=\lim_{x\to\frac{-1}{2}}\frac{(2x+1)(4x^2-2x+1)}{2x+1}=3$

$e)\ \lim_{x\to 3}\frac{\frac{1}{x}-\frac{1}{3}}{x^3-27}=\lim_{x\to 3}\frac{\frac{3-x}{3x}}{(x-3)(x^2+3x+9)}=\lim_{x\to 3}\frac{-\frac{1}{3x}}{(x^2+3x+9)}=\frac{-1}{243}$

$f)\ \lim_{x\to 2}\frac{x^3-8}{x^2-4}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}=\lim_{x\to 2}\frac{x^2+2x+4}{x+2}=3$

2. If $\lim_{x\to a}\frac{a^3-5a^2x}{a^2-25x^2}=\lim_{x\to 1}\frac{x^4-1}{x^3-1}$, then find a.

Solution:

$$\lim_{x\to a}\frac{a^3-5a^2x}{a^2-25x^2}=\lim_{x\to 1}\frac{x^4-1}{x^3-1}\Rightarrow\lim_{x\to a}\frac{a^2(a-5x)}{(a-5x)(a+5x)}=\lim_{x\to 1}\frac{(x-1)(x+1)(x^2+1)}{(x-1)(x^2+x+1)}$$

$$\Rightarrow\lim_{x\to a}\frac{a^2}{a+5x}=\lim_{x\to 1}\frac{(x+1)(x^2+1)}{x^2+x+1}\Rightarrow\frac{a}{6}=\frac{4}{3}\Rightarrow a=8$$

iii) By Rationalization: This method is used to evaluate when the limit problem contains at least one radical or nth root as sum or difference like $\sqrt{f}\pm\sqrt{g}$ **or** $\sqrt[3]{f}\pm\sqrt[3]{g}$ **or** $\sqrt{f}\pm g$. In such case, the limit is evaluated by rationalizing the radical term from the numerator or the denominator so that by cancelling a term that creates a problem to evaluate the limit.

Examples:

$a)\ \lim_{x\to 2}\frac{\sqrt{x+7}-3}{x-2}$ $\qquad b)\ \lim_{x\to 4}\frac{x-4}{\sqrt{x^2+9}-5}$ $\qquad c)\ \lim_{x\to 0}\frac{\sqrt{2+3x}-\sqrt{2-3x}}{x}$

$d)\ \lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x}$ $\qquad e)\ \lim_{x\to 9}\frac{\sqrt{x-3\sqrt{x}+4}-2}{\sqrt{x}-3}$ $\qquad f)\ \lim_{x\to\infty}(\sqrt{x^2+x+1}-x)$

Solution:

a) Here, if we try to evaluate by direct substitution, the term $x-2$ in the denominator leads to undefined result. Thus, we need a factor $x-2$ in the numerator so that to cancel it. So, rationalize $\sqrt{x+7}-3$ to get the factor.

$$\lim_{x\to 2}\frac{\sqrt{x+7}-3}{x-2}=\lim_{x\to 2}\frac{(\sqrt{x+7}-3)}{x-2}\left(\frac{\sqrt{x+7}+3}{\sqrt{x+7}+3}\right)=\lim_{x\to 2}\frac{x+7-9}{(x-2)\sqrt{x+7}+3}$$

$$=\lim_{x\to 2}\frac{x-2}{(x-2)\sqrt{x+7}+3}=\lim_{x\to 2}\frac{1}{\sqrt{x+7}+3}=\frac{1}{6}$$

b) $$\lim_{x\to 4}\frac{x-4}{\sqrt{x^2+9}-5}=\frac{x-4}{\sqrt{x^2+9}-5}\left(\frac{\sqrt{x^2+9}+5}{\sqrt{x^2+9}+5}\right)=\lim_{x\to 4}\frac{(x-4)(\sqrt{x^2+9}+5)}{x^2-16}$$

$$=\lim_{x\to 4}\frac{(x-4)(\sqrt{x^2+9}+5)}{(x-4)(x+4)}=\lim_{x\to 4}\frac{\sqrt{x^2+9}+5}{x+4}=\frac{10}{8}=\frac{5}{4}$$

c) $$\lim_{x\to 0}\frac{\sqrt{2+3x}-\sqrt{2-3x}}{x}=\lim_{x\to 0}\left(\frac{\sqrt{2+3x}-\sqrt{2-3x}}{x}\right)\left(\frac{\sqrt{2+3x}+\sqrt{2-3x}}{\sqrt{2+3x}+\sqrt{2-3x}}\right)$$

$$=\lim_{x\to 0}\frac{2+3x-(2-3x)}{x(\sqrt{2+3x}+\sqrt{2-3x})}=\lim_{x\to 0}\frac{6}{\sqrt{2+3x}+\sqrt{2-3x}}=\frac{3}{\sqrt{2}}$$

d) $$\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-2x}=\lim_{x\to 2}\frac{(\sqrt{x^2+5}-3)}{x^2-2x}\left(\frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3}\right)=\lim_{x\to 2}\frac{x^2-4}{(x^2-2x)(\sqrt{x^2+5}+3)}$$

$$=\lim_{x\to 2}\frac{(x-2)(x+2)}{x(x-2)(\sqrt{x^2+5}+3)}=\lim_{x\to 2}\frac{x+2}{x(\sqrt{x^2+5}+3)}=\frac{1}{3}$$

e) $$\lim_{x\to 3}\frac{\sqrt{x-3\sqrt{x}+4}-2}{\sqrt{x}-3}=\lim_{x\to 9}\frac{\sqrt{x}(\sqrt{x}-3)}{(\sqrt{x}-3)(\sqrt{x-3\sqrt{x}+4}+2)}=\frac{3}{4}$$

f) $$\lim_{x\to\infty}(\sqrt{x^2+x+1}-x)=\lim_{x\to\infty}\frac{(\sqrt{x^2+x+1}-x)(\sqrt{x^2+x+1}+x)}{\sqrt{x^2+x+1}+x}$$

$$=\lim_{x\to\infty}\frac{x+1}{\sqrt{x^2+x+1}+x}=\frac{1}{2}$$

iv) By Using Important limit Theorems: The following limits are the basic tools in the evaluation of limits. So, please bear them in mind for the next use.

$$i)\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad ii)\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=\lim_{x\to 0^{+}}(1+x)^{\frac{1}{x}}=e \qquad iii)\lim_{n\to\infty}x^{n}=0,|x|<1$$

Remarks: $i)\ \lim_{x\to 0}\frac{\sin ax}{bx}=\frac{a}{b} \qquad ii)\ \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx+c}=e^{ab} \qquad iii)\ \lim_{x\to 0^{+}}(1+kx)^{\frac{1}{x}}=e^{k}$

Examples:

1. Evaluate

$$a)\ \lim_{x\to\infty}\left(\frac{x+3}{x}\right)^{x} \qquad b)\ \lim_{x\to\infty}\left(\frac{2+x}{x}\right)^{3x+5} \qquad c)\ \lim_{x\to 0}\frac{\sin 5x}{x}$$

$$d)\ \lim_{x\to 0}\frac{7x^{2}+\tan^{2}3x}{x^{2}} \qquad f)\ \lim_{x\to 0}\frac{\sin^{3}2x}{x^{3}} \qquad f)\ \lim_{x\to\infty}\left(\frac{4^{x}+5^{x}}{7^{x}}\right)$$

Solution:

$$a)\ \lim_{x\to\infty}\left(\tfrac{x+3}{x}\right)^{x}=\lim_{x\to\infty}\left(1+\tfrac{3}{x}\right)^{x}=e^{3}$$

$$b)\ \lim_{x\to\infty}\left(\tfrac{2+x}{x}\right)^{3x+5}=\lim_{x\to\infty}\left(1+\tfrac{2}{x}\right)^{3x+5}=e^{6}$$

$$c)\ \lim_{x\to 0}\frac{\sin 5x}{x}=5\lim_{x\to 0}\frac{\sin 5x}{5x}=5,\quad (\because\ \lim_{x\to 0}\frac{\sin ax}{ax}=1)$$

$$d)\ \lim_{x\to 0}\frac{7x^{2}+\tan^{2}3x}{x^{2}}=7+9\lim_{x\to 0}\left(\frac{\tan 3x}{3x}\right)^{2}=7+9\left(\lim_{x\to 0}\frac{\sin 3x}{3x}\right)^{2}\left(\lim_{x\to 0}\frac{1}{\cos 3x}\right)=16$$

$$e)\ \lim_{x\to 0}\frac{\sin^{3}2x}{x^{3}}=\lim_{x\to 0}\frac{8\sin^{3}2x}{8x^{3}}=8\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)^{3}=8\left(\lim_{x\to 0}\frac{\sin 2x}{2x}\right)^{3}=8$$

2. Evaluate the limit: $\lim_{x\to\infty}x\ln\left(\frac{x+2}{x-2}\right)$

Solution: Rearrange using logarithm property and divide the numerator and denominator by x. Then, insert the limit inside the logarithm as follow.

$$\lim_{x\to\infty}x\ln\left(\frac{x+2}{x-2}\right)=\lim_{x\to\infty}\ln\left(\frac{x+2}{x-2}\right)^{x}=\ln\lim_{x\to\infty}\frac{\left(1+\frac{2}{x}\right)^{x}}{\left(1-\frac{2}{x}\right)^{x}}=\ln\frac{e^{2}}{e^{-2}}=\ln e^{4}=4$$

3. If $\lim_{x\to\infty}\left(\frac{x+c}{x-5}\right)^{2x}=e^{36}$, then find the constant c.

Solution:

$$\lim_{x\to\infty}\left(\frac{x+c}{x-5}\right)^{2x}=\lim_{x\to\infty}\left(\frac{1+c/x}{1-5/x}\right)^{2x}=\frac{\lim_{x\to\infty}(1+c/x)^{2x}}{\lim_{x\to\infty}(1-5/x)^{2x}}=\frac{e^{2c}}{e^{-10}}=e^{2c+10}.$$

Hence, $e^{2c+10}=e^{36}\Rightarrow 2c+10=36\Rightarrow c=13$

4. Evaluate the following limits

$a)\lim_{x\to0}\frac{e^{2x}}{(x^2-2x)\csc 3x}$ $b)\ \lim_{x\to1}\frac{x^5-x^2-4x+4}{x^2-1}$ $c)\lim_{x\to2\pi}\frac{(x^2+2x)\sin x}{x-2\pi}$

Solution: Rearrange to use the important limit.

a) Use the trig relation, $\csc kx=\frac{1}{\sin kx}$ **or** $\sin kx=\frac{1}{\csc kx}$.

That is $\lim_{x\to0}\frac{e^{2x}}{(x^2-2x)\csc 3x}=\frac{e^{2x}}{x(x-2)}\cdot\frac{1}{\csc 3x}=\lim_{x\to0}\frac{e^{2x}}{x-2}\cdot\lim_{x\to0}\frac{\sin 3x}{x}=-\frac{3}{2}$

b) Since the numerator and denominator are zero at $x=1, x-1$ is a factor of both.

So, first divide x^5-x^2-4x+4 by $x-1$ to get the factorization.

That is by long division, $x^5-x^2-4x+4=(x-1)(x^4+x^3+x^2-4)$.

So, $\lim_{x\to1}\frac{x^5-x^2-4x+4}{x^2-1}=\lim_{x\to1}\frac{(x-1)(x^4+x^3+x^2-4)}{(x-1)(x+1)}=-\frac{1}{2}$

c) Use the substitution $y=x-2\pi, y\to0$ as $x\to2\pi$.

v) By dividing the numerator and denominator by the highest power of the numerator. For limits involving rational functions, divide the numerator and the denominator by the highest power of the numerator and evaluate the limit using the concepts under limit at infinity.

Examples: Evaluate

$a)\ \lim_{x\to\infty}\frac{5x^3+3x-4}{x^3+1}$ $b)\ \lim_{x\to\infty}\frac{x^2+3}{x^5+1}$ $c)\ \lim_{x\to\infty}\frac{3x}{\sqrt{16x^2+3}}$

$d)\lim_{x\to\infty}(\sqrt{x^2+x+1}-x)$ $e)\ \lim_{x\to-\infty}(\sqrt{x^2+3x+1}+x)$ $f)\lim_{x\to\infty}(\sqrt{9x^2+x}-3x)$

Solution:

a) $\lim\limits_{x\to\infty}\dfrac{5x^3+3x-4}{x^3+1}=\lim\limits_{x\to\infty}\dfrac{5+3/x^2-4/x^3}{1+1/x^2}=5$

b) $\lim\limits_{x\to\infty}\dfrac{x^2+3}{x^5+1}=\lim\limits_{x\to\infty}\dfrac{1+3/x^2}{x^3+1/x^3}=0$

c) $\lim\limits_{x\to\infty}\dfrac{3x}{\sqrt{16x^2+3}}=\lim\limits_{x\to\infty}\dfrac{3}{\sqrt{16+3/x}}=\dfrac{3}{4}$

d) $\lim\limits_{x\to\infty}(\sqrt{x^2+x+1}-x)=\lim\limits_{x\to\infty}\dfrac{(\sqrt{x^2+x+1}-x)(\sqrt{x^2+x+1}+x)}{\sqrt{x^2+x+1}+x}=\dfrac{1}{2}$

e) $\lim\limits_{x\to-\infty}(\sqrt{x^2+3x+1}+x)=\lim\limits_{x\to-\infty}\dfrac{(\sqrt{x^2+3x+1}+x)(\sqrt{x^2+3x+1}-x)}{\sqrt{x^2+3x+1}-x}$

$$=\lim_{x\to-\infty}\frac{3x+1}{\sqrt{x^2+3x+1}-x}=-\frac{3}{2}$$

f) $\lim\limits_{x\to\infty}(\sqrt{9x^2+x}-3x)=\lim\limits_{x\to\infty}\dfrac{(\sqrt{9x^2+x}-3x)(\sqrt{9x^2+x}+3x)}{\sqrt{9x^2+x}+3x}=\lim\limits_{x\to\infty}\dfrac{x}{\sqrt{9x^2+x}+3x}=\dfrac{1}{6}$

vi) By Squeezing Theorem (Squeezing or sandwich theorem).

Given three functions f,g,h such that $f(x)\le h(x)\le g(x)$ for every x in open interval containing a, not necessarily at a itself.

If $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}g(x)=L$, then $\lim\limits_{x\to a}h(x)=L$.

Examples: Evaluate the following limits using Squeezing Theorem.

a) $\lim\limits_{x\to 0}x^2\sin\left(\dfrac{1}{x}\right)$ b) $\lim\limits_{x\to\infty}\dfrac{x^2\cos(x^3)}{x^4+1}$ c) $\lim\limits_{x\to\infty}\dfrac{x^2(\sin x+\cos^3 x)}{(x^2+1)(x-1)}$

Solution: Recall that for any real number x, $-1\le\sin x\le 1$. So,

a) $-1\le\sin\left(\dfrac{1}{x}\right)\le 1\Rightarrow -x^2\le x^2\sin\left(\dfrac{1}{x}\right)\le x^2$. Besides, $\lim\limits_{x\to 0}-x^2=\lim\limits_{x\to 0}x^2=0$.

Hence, by Squeezing Theorem, $\lim\limits_{x\to 0}x^2\sin\left(\dfrac{1}{x}\right)=0$.

b) Recall that for any function f(x), $-1 \le \cos f(x) \le 1$.

So, $-1 \le \cos(x^3) \le 1 \Rightarrow -x^2 \le x^2\cos(x^3) \le x^2$

$$\Rightarrow \frac{-x^2}{x^4+1} \le \frac{x^2\cos(x^3)}{x^4+1} \le \frac{x^2}{x^4+1}$$

But if we take $f(x)=\frac{-x^2}{x^4+1}, g(x)=\frac{x^2}{x^4+1}, h(x)=\frac{x^2\cos(x^3)}{x^4+1}$ in Squeezing Theorem, we have $f(x) \le h(x) \le g(x)$ where $\lim_{x\to\infty}\frac{-x^2}{x^4+1}=0=\lim_{x\to\infty}\frac{x^2}{x^4+1}$.

Therefore, $\lim_{x\to\infty}\frac{x^2\cos(x^3)}{x^4+1}=0$.

c) $-1 \le \sin x \le 1, -1 \le \cos^3 x \le 1 \Rightarrow -2 \le \sin x + \cos^3 x \le 2$

$$\Rightarrow \frac{-2x^2}{(x^2+1)(x-1)} \le \frac{x^2(\sin x+\cos^3 x)}{(x^2+1)(x-1)} \le \frac{2x^2}{(x^2+1)(x-1)}$$

But $\lim_{x\to\infty}\frac{-2x^2}{(x^2+1)(x-1)}=\lim_{x\to\infty}\frac{2x^2}{(x^2+1)(x-1)}=0 \Rightarrow \lim_{x\to\infty}\frac{x^2(\sin x+\cos^3 x)}{(x^2+1)(x-1)}=0$

4.4 Applications of Limits

Asymptotes: We know that vertical, horizontal and oblique asymptotes are useful guides in drawing the graphs of rational functions but the determination of these asymptotes is a little bit complicated. However, such difficulties are avoided if one knows the limit approach of determining asymptotes.

i) **Vertical asymptotes:** The line $x = a$ is said to be vertical asymptote to the graph of f if one of the following is true:

$$\lim_{x\to a^-} f(x) = \pm\infty,\ \lim_{x\to a^+} f(x) = \pm\infty,\ \lim_{x\to a} f(x) = \pm\infty$$

ii) **Horizontal asymptotes:** The line $y = k$ is a horizontal asymptote to the graph of f if either $\lim_{x\to -\infty} f(x) = k$ **or** $\lim_{x\to +\infty} f(x) = k$.

iii) **Oblique asymptotes:** The line $y = ax + b$ is said to be oblique asymptote to the graph of f if $\lim_{x\to a}[f(x) - ax - b] = 0$.

In such case, the asymptote is obtained by $a = \lim_{x\to\infty} \frac{f(x)}{x}$, $b = \lim_{x\to\infty}[f(x) - ax]$.

Examples: Find the *vertical*, *horizontal* and *oblique* asymptotes of

a) $f(x) = \frac{6x}{x-2}$ *b)* $f(x) = \frac{2x^3 - 3x^2}{x^2+1}$ *c)* $f(x) = \frac{\sqrt{9x^2+7}}{x-4}$

d) $f(x) = x + e^{-x}\sin x$ *e)* $f(x) = \ln(x-2)$ *f)* $f(x) = \frac{2x+3}{\sqrt{x^2-2x-3}}$

Solution:

a) $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{6x}{x-2} = -\infty$, $\lim_{x\to 2^+} \frac{6x}{x-2} = +\infty \Rightarrow x = 2$ is vertical asymptote.

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{6x}{x-2} = 3 \Rightarrow y = 3$ is horizontal asymptote.

Since $\deg p(x) = \deg q(x)$, there is no oblique asyptote.

b) Since $\deg p(x) = 1 + \deg q(x)$, we have oblique asymptote of $y = ax + b$.

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{2x^3 - 3x^2}{x^3 + x} = 2,\ b = \lim_{x\to\infty}[f(x) - ax] = \lim_{x\to\infty} \frac{-3x^2 - 2x}{x^2+1} = -3$$

Hence, the oblique asymptote is $y = 2x - 3$.

c) $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} \frac{\sqrt{9x^2+7}}{x-4} = -\infty$; $\lim_{x\to 4^+} \frac{\sqrt{9x^2+7}}{x-4} = +\infty$.

Hence, $x = 4$ is vertical asyptote.

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{\sqrt{9x^2+7}}{x-4} = \lim_{x\to\pm\infty} \frac{|x|\sqrt{x^2(9+7/x^2)}}{x-4} = \lim_{x\to\pm\infty} \frac{|x|\sqrt{9+7/x^2}}{x-4} = \pm 3$

Hence, $y = 3$ and $y = -3$ are horizontal asymptotes

Since $\deg p(x) = \deg q(x)$, there is no oblique asyptote.

d) Here, suppose there is an oblique asymptote of $y = ax + b$.

Then, $a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{x + e^{-x}\sin x}{x} = 1$,

$b = \lim_{x\to\infty}[f(x) - ax] = \lim_{x\to\infty}\left(x + e^{-x}\sin x - x\right) = \lim_{x\to\infty} e^{-x}\sin x = 0$.

Hence, the oblique asymptoteis $y = x$.

e) Since $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \ln(x-2) = -\infty$, $x = 2$ is a vertical asymptote.

But $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \ln(x-2) = \infty$, there is no horizontal asymptote.

f) Here, $x^2 - 2x - 3 = 0 \Rightarrow (x-3)(x+1) = 0 \Rightarrow x = 3, x = -1$ .So.

Then, $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \frac{2x+3}{\sqrt{x^2-2x-3}} = \infty$, $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{2x+3}{\sqrt{x^2-2x-3}} = +\infty$,

Hence, the lines $x = 3$ and $x = -1$ are vertical asymptotes.

Besides, Then, $\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{2x+3}{\sqrt{x^2-2x-3}} = \lim_{x\to\pm\infty} \frac{2x+3}{|x|\sqrt{(1-2/x-3/x^2)}} = \pm 2$.

Therefore, the lines $y = \pm 2$ are horizontal asymptotes.

4.5 Continuity Of Functions

Definition: A function is said to be continuous at a point $x = a$ if and only if the following three conditions are satisfied:

i) $f(a)$ is defined (a is in the domain of f)

ii) $\lim\limits_{x \to a} f(x)$ exists

iii) $\lim\limits_{x \to a} f(x) = f(a)$

If one of these conditions is not satisfied, we say that the function is discontinuous at $x = a$ and the point $x = a$ is called point of discontinuity.

Examples:

Check whether the functions are continuous or not at the indicated point.

a) $f(x) = \begin{cases} x^2 + 1; x \le 2 \\ 3x - 2; x > 2 \end{cases}$ at $x = 2$ b) $f(x) = \begin{cases} \cos x; x < 0 \\ 0\ ; x = 0 \\ 1 - x^2; x > 0 \end{cases}$ at $x = 0$

Solution:

a) $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x^2 + 1) = 5;\ \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (3x - 1) = 5$

$$\Rightarrow \lim_{x \to 2} f(x) = 5;\ f(2) = 5$$

Thus, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2} f(x) = f(2) = 5$

Here, all the three conditions are satisfied. Therefore, f is continuous at $x = 2$.

b) $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (\cos x) = 1;\ \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} (1 - x^2) = 1$

$$\Rightarrow \lim_{x \to 0} f(x) = 1;\ f(0) = 0$$

Here, $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0} f(x) = 1 \ne f(0) = 0$

Here, the third condition is not satisfied. Therefore, f is discontinuous at $x = 0$.

4.6 One Sided Continuity

I) **Continuity From the left:** A function f is said to be continuous from the left at $x = a$ if and only if $\lim_{x\to a^-} f(x) = f(a)$.

II) **Continuity From the right:** A function f is said to be continuous from the right at $x = a$ if and only if $\lim_{x\to a^+} f(x) = f(a)$.

Examples: Show that

a) The function $f(x) = \begin{cases} x^2+1; x \le 0 \\ x-1; x > 0 \end{cases}$ is continuous from the left at $x = 0$ but not from the right.

b) The function $f(x) = \begin{cases} \frac{x^2-9}{x-3}; x < 3 \\ x+1; x \ge 3 \end{cases}$ is continuous from the right at $x = 3$ but not from the left.

c) The function $f(x) = \begin{cases} \frac{7x - \tan 3x}{x}; x \ne 0 \\ x+4;\ x = 0 \end{cases}$ is continuous both from the right and from the left at $x = 0$.

Solution:

a) $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^2+1) = 1 = f(0)$; but $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x-1) = -1 \ne f(0)$.

b) $\lim_{x\to 3^+} f(x) = 4 = f(3)$; but $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-}\frac{x^2-9}{x-3} = \lim_{x\to 3^-}(x+3) = 6 \ne f(3)$.

c) Here, $f(0) = 4$, $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0}\frac{7x - \tan 3x}{x}$

$$= \lim_{x\to 0}(7 - \frac{\tan 3x}{x}) = 7 - 3 = 4 = f(0)$$

Composition and Limits: Continuity rule for limit:

If a function f is continuous at b and $\lim_{x\to a} g(x) = b$, then we have

$$\lim_{x\to a}(f\circ g)(x) = \lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b).$$

Examples:

1. Evaluate $a)\ \lim_{x\to 1}\cos^{-1}\left(\frac{\sqrt{x}-1}{x-1}\right)$ $\quad b)\ \lim_{x\to\frac{\pi}{2}}\sin\left(\frac{1}{3}(2x+\sin 2x)\right)$

Solution: a) Since $f(x)=\cos^{-1}x$ is continuous at $x=1$, by the above theorem, we have $\lim_{x\to 1}\cos^{-1}\left(\frac{\sqrt{x}-1}{x-1}\right) = \cos^{-1}\left(\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$.

b) Since sine function is continuous everywhere, by the above property, we have

$$\lim_{x\to\frac{\pi}{2}}\sin\left(\frac{1}{3}(2x+\sin 2x)\right) = \sin\left(\frac{1}{3}\lim_{x\to\frac{\pi}{2}}(2x+\sin 2x)\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}.$$

2. Let $f(x)=x^2+e^{x+8}, g(x)=\ln x, h(x)=\sqrt[3]{x}$, Then, evaluate $\lim_{x\to 0} h(g(f(x)))$

Solution: Since $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^2+e^{x+8}) = e^8$ and $g(x)=\ln x$ continuous at $x=e^8$, use the above composition property as follow:

$$\lim_{x\to 0} h(g(f(x))) = h(\lim_{x\to 0} g(f(x))) = h(g(\lim_{x\to 0} f(x)))$$
$$= h(g(e^8)) = h(\ln e^8) = h(8) = \sqrt[3]{8} = 2$$

3. Suppose $f: R\to R$ is a continuous function such that $f(2)=\sqrt{2}$. Then, evaluate $\lim_{x\to 0} f\left(\frac{\sqrt{1+2x^2}-\sqrt{1-2x^2}}{x^2}\right)$.

Solution: Since f is continuous on R, it is also continuous at $L=\lim_{x\to 0} g(x)$.

But $L=\lim_{x\to 0} g(x) = \lim_{x\to 0}\frac{\sqrt{1+2x^2}-\sqrt{1-2x^2}}{x^2}\left(\frac{\sqrt{1+2x^2}+\sqrt{1-2x^2}}{\sqrt{1+2x^2}+\sqrt{1-2x^2}}\right) = 2$

Hence, $\lim_{x\to 0} f\left(\frac{\sqrt{1+2x^2}-\sqrt{1-2x^2}}{x^2}\right) = f(L) = f(L) = f(2) = \sqrt{2}$

4.7 Re-defining Discontinuous Functions to be Continuous

Definition: We say that a function f has removable discontinuity at $x=a$ if $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a} f(x) \neq f(a)$. In this case, we say that the point a is *removable* discontinuity. On the other hand, if $\lim_{x\to a} f(x)$ does not exist, then we say that the point a is *non-removable* discontinuity. A function with removable discontinuity at $x=a$ can be **re-defined** to be continuous at a. This procedure is known as *re-defining technique*. Suppose $\lim_{x\to a} f(x) = L$, but $\lim_{x\to 2} f(x) \neq f(a)$.

Thus, f can be *re-defined* as $f(x)=\begin{cases} f(x); x\neq a \\ L; x=a \end{cases}$

Now, f becomes continuous at $x=a$.

Remark: In general, if the function f is continuous at a point $x=a$, then we have $\lim_{x\to a^-} f(x)=f(a), \lim_{x\to a^+} f(x)=f(a), \lim_{x\to a} f(x)=f(a)$. This result is useful in finding constants or in re-defining a function by selecting appropriate constants at a given point.

Examples:

1. Re-define f (if possible) to be continuous at the indicated point.

a) $f(x)=\dfrac{\tan 3x}{x}$ at $x=0$ b) $f(x)=\begin{cases} 5x-1; x<2 \\ x^3+1; x>2 \end{cases}$ at $x=2$

Solution:

a) Here, $\lim_{x\to 0} f(x) = \lim_{x\to 0}\dfrac{\tan 3x}{x} = 3$. This means $\lim_{x\to 0} f(x)$ exists but $f(0)$ is not defined. Hence, the re-defined form of f is $f(x)=\begin{cases} \dfrac{\tan 3x}{x}; x\neq 0 \\ 3; x=0 \end{cases}$

Now, observe that f becomes continuous at $x=0$.

b) Here, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(5x-1)=9; \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(x^3+1)=9$.

Since $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x), \lim_{x\to 2} f(x)$ exists but $f(2)$ is not defined at $x=2$.

2. Find the constants c and k o that

a) $f(x)=\begin{cases} cx^2+2x, & x\le 2 \\ x^3-cx, & x>2 \end{cases}$ will be continuous at $x=2$.

b) $f(x)=\begin{cases} k(x-1)^2, & x<3 \\ 4x+2k, & x\ge 3 \end{cases}$ will be continuous at $x=3$

c) $f(x)=\begin{cases} \sin(2k+x), & x\le 0 \\ 1-x, & x>0 \end{cases}$ will be continuous at $x=0$

d) $f(x)=\begin{cases} x, & x\le 1 \\ cx+k, & 1<x<4 \\ -2x, & x\ge 4 \end{cases}$ will be continuous for all real numbers

Solution: We use the general conditions of continuity stated above.

a) Here, $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^+}(x^3-cx)=8-2c$ and $f(2)=4c+4$.

To be continuous at $x=2$, apply continuity conditions as follow:

$$\lim_{x\to 2^+} f(x)=f(2)\Rightarrow 8-2c=4c+4\Rightarrow 6c=4\Rightarrow c=\tfrac{2}{3}$$

b) Here, $\lim\limits_{x\to 3^-} f(x)=\lim\limits_{x\to 3^-} k(x-1)^2=4k$ and $f(3)=12+2k$.

To be continuous at $x=3$, apply continuity conditions as follow:

$$\lim_{x\to 3^-} f(x)=f(3)\Rightarrow 4k=12+2k\Rightarrow 3k=12\Rightarrow k=4.$$

c) Here, $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^-}\sin(2k+x)=\sin 2k$, $\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}(1-x)=1$.

To be continuous at $x=0$, we must have $\lim\limits_{x\to 0^-} f(x)=\lim\limits_{x\to 0^+} f(x)=f(0)$.

That is $\sin 2k=1\Rightarrow 2k=\sin^{-1}(1)\Rightarrow 2k=\frac{\pi}{2}\Rightarrow k=\frac{\pi}{4}$.

d) Since we need continuity for all real numbers, we can analyze only at the boundary points $x=1, x=4$ where discontinuity is expected.

At $x=1$, $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} x=1$, $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+}(cx+k)=c+k$.

For continuity at $x=1$, $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)\Rightarrow c+k=1$.

At $x=4$, $\lim\limits_{x\to 4^-} f(x)=\lim\limits_{x\to 4^-}(cx+k)=4c+k$, $\lim\limits_{x\to 4^+} f(x)=\lim\limits_{x\to 4^+}(-2x)=-8$.

For continuity at $x=4$, $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) \Rightarrow 4c+k=-8$.

Hence, we get the system and its solution; $\begin{cases} c+k=1 \\ 4c+k=-8 \end{cases} \Rightarrow \begin{cases} c=-3 \\ k=4 \end{cases}$

3. Find the value(s) of k so that the function will be continuous at $x=0$.

a) $f(x) = \begin{cases} \dfrac{6x+\tan(kx)}{x}, & x \neq 0 \\ k^2, & x=0 \end{cases}$ b) $f(x) = \begin{cases} \dfrac{\tan(kx)}{x}, & x<0 \\ 3x+2k^2, & x \geq 0 \end{cases}$

Solution: To be continuous at $x=0$,

a) $\lim_{x \to 0} f(x) = f(0) \Rightarrow \lim_{x \to 0}\left(\dfrac{6x+\tan(kx)}{x}\right) = k^2 \Rightarrow k^2 = 6+k$

$$\Rightarrow k^2 - k - 6 = 0 \Rightarrow k = 3, -2.$$

b) $\lim_{x \to 0} f(x) = f(0) \Rightarrow \lim_{x \to 0} \dfrac{\tan(kx)}{x} = 2k^2 \Rightarrow k = 2k^2 \Rightarrow k = 0, \dfrac{1}{2}$

4. Find a, b and c so that f will be continuous at every real number in R.

a) $f(x) = \begin{cases} 6-x^2, & x \leq c \\ x, & x>c \end{cases}$ b) $f(x) = \begin{cases} \dfrac{x^3-a^2x}{x-a}, & x \neq a \\ 18, & x=a \end{cases}$

c) $f(x) = \begin{cases} x^2+7, & x<a \\ 21-5x, & a \leq x \leq b \\ 5x-19, & x>b \end{cases}$ d) $f(x) = \begin{cases} 3x-2, & x<2 \\ ax^2-bx+3, & 2 \leq x<3 \\ 2x-a+b; & x \geq 3 \end{cases}$

Solution: We need the functions to be continuous on R (at any real number). Particularly, we need the function to be continuous at the breaking points.

a) To be continuous at $x=c$, by continuity condition, we must have

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \Rightarrow 6-c^2 = c \Rightarrow c^2+c-6=0 \Rightarrow c=-3, 2.$$

b) Here, $\lim_{x \to a} f(x) = \lim_{x \to a} \dfrac{x^3-a^2x}{x-a} = \lim_{x \to a} \dfrac{x(x-a)(x+a)}{x-a} = \lim_{x \to a} x(x+a) = 2a^2$.

To be continuous at $x=a$, $\lim_{x \to a} f(x) = f(a) \Rightarrow 2a^2 = 18 \Rightarrow a^2 = 9 \Rightarrow a = \pm 3$

c) To be continuous at $x = a$,

$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) \Rightarrow a^2 + 7 = 21 - 5a \Rightarrow a^2 + 5a - 14 = 0 \Rightarrow a = 2, -7$

Similarly, at $x = b$, $\lim_{x\to b^-} f(x) = \lim_{x\to b^+} f(x) \Rightarrow 21 - 5b = 5b - 19 \Rightarrow b = 4$

d) To be continuous at $x = 2$ and $x = 3$, solve the systems as follow:

$$\begin{cases} \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) \Rightarrow 4a - 2b + 3 = 4 \Rightarrow 4a - 2b = 1 \\ \lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \Rightarrow 9a - 3b + 3 = 6 - a + b \Rightarrow 10a - 4b = 3 \end{cases} \Rightarrow a = b = \tfrac{1}{2}$$

5. Suppose $\lim_{x\to 1} \frac{3f(x) + 4g(x)}{x^2 + 1} = 15$, $f(1) = 6$ where f and g are continuous in all real numbers. Then, find the value of $g(1)$.

Solution: Since f and g are continuous, we have

$$\lim_{x\to 1} \frac{3f(x) + 4g(x)}{x^2 + 1} = 15 \Rightarrow \frac{3\lim_{x\to 1} f(x) + 4\lim_{x\to 1} g(x)}{\lim_{x\to 1}(x^2 + 1)} = 15 \Rightarrow \frac{18 + 4g(1)}{2} = 15$$

$$\Rightarrow 18 + 4g(1) = 30 \Rightarrow 4g(1) = 12 \Rightarrow g(1) = 3$$

Composition and Limits: If a function f is continuous at b and $\lim_{x\to a} g(x) = b$, then we have $\lim_{x\to a}(f \circ g)(x) = \lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$.

Examples: 1. Evaluate $a)$ $\lim_{x\to 1} \cos^{-1}\left(\frac{\sqrt{x} - 1}{x - 1}\right)$ $b)$ $\lim_{x\to \frac{\pi}{2}} \sin\left(\frac{1}{3}(2x + \sin 2x)\right)$

Solution:

a) Since $f(x) = \cos^{-1} x$ is continuous at $x = 1$, by the above theorem, we have

$$\lim_{x\to 1} \cos^{-1}\left(\frac{\sqrt{x} - 1}{x - 1}\right) = \cos^{-1}\left(\lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

2. Let $f(x) = x^2 + e^{x+8}, g(x) = \ln x, h(x) = \sqrt[3]{x}$, Then, evaluate $\lim_{x\to 0} h(g(f(x)))$.

Solution: Since $\lim_{x\to 0} f(x) = \lim_{x\to 0}(x^2 + e^{x+8}) = e^8$ and $g(x) = \ln x$ is continuous at $x = e^8$, $\lim_{x\to 0} h(g(f(x))) = h(\lim_{x\to 0} g(f(x))) = h(g(e^8)) = h(\ln(e^8)) = h(8) = \sqrt[3]{8} = 2$

4.8 Intermediate Value Theorem (IVT)

Suppose f is continuous on $[a,b]$ and c is any number between $f(a)$ and $f(b)$

...ive. That is $f(a) \le c \le f(b)$ or $f(b) \le c \le f(...$

Review Problems on Chapter-4

1. Evaluate the following limits....

a) $\lim\limits_{x \to 3} \dfrac{3-x}{2-\sqrt{x^2-5}}$ b) $\lim\limits_{h \to 3} \dfrac{1-\sqrt{h-2}}{h-3}$ c) $\lim\limits_{h \to 0} \dfrac{\sqrt{4+h}-2}{h}$

d) $\lim\limits_{x \to 0} \dfrac{\sqrt{2+3x}-\sqrt{2-3x}}{x}$ e) $\lim\limits_{h \to 0} \dfrac{\sqrt[4]{1+h}-1}{h}$ f) $\lim\limits_{x \to 0} \dfrac{\sqrt{1+3xe^x}-\sqrt{1-3xe^x}}{x}$

g) $\lim\limits_{x \to \infty} (\sqrt{x^2+3x}-x)$ h) $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^3}}{\sqrt{x^3+\sqrt{x^3+\sqrt{x^3+3}}}}$ i) $\lim\limits_{h \to 0} \left(\dfrac{x^2-2x^3}{x^2}\right)^{[illegible]}$

Answer : a) $\frac{2}{3}$ b) $-\frac{1}{2}$ c) $\frac{1}{4}$ d) $\frac{3\sqrt{2}}{2}$ e) $\frac{1}{4}$ f) 3 g) $\frac{3}{2}$ h) 1 i) $e^{[illegible]}$

2. Evaluate the following limits

a) $\lim\limits_{x \to 0} \dfrac{1-\cos 5x}{\cos 7x - 1}$ b) $\lim\limits_{h \to 3} \dfrac{1}{x-1}\left(\dfrac{1}{x+3}-\dfrac{2x}{3x+5}\right)$ c) $\lim\limits_{x \to 0} \dfrac{x^2}{1-\cos x}$ d) $\lim\limits_{x \to +\infty} x\sin\dfrac{1}{x}$

Answer : a) $-\frac{25}{49}$ b) $\frac{1}{32}$ c) 2 d) 1

3. Find the horizontal asymptotes of the following functions.

a) $f(x)=\dfrac{\sqrt{5x^2-2}}{x+3}$ b) $f(x)=\dfrac{\sqrt{3x^4+x}}{x^2-8}$ c) $f(x)=\dfrac{2x+3}{\sqrt{x^2-2x-3}}$

d*) $f(x)=\dfrac{\sqrt[3]{8x^3-1}}{\sqrt{9x^2-7}}$ e*) $f(x)=\dfrac{3e^{-x}+8e^{2x}}{5e^{-x}+2e^x}$

Answer : a) $y=\sqrt{5}$ b) $y=\sqrt{3}$ c) $y=\pm 2$ d) $y=\pm\frac{2}{3}$ e) $y=\frac{3}{5}, 4$

4. Find the vertical asymptotes of the following functions.

a*) $f(x)=\dfrac{\sin(x^2-2x)}{x^3-4x}$ b) $f(x)=\dfrac{x^2}{e^{2x-1}-e^x}$ **Answer** : a) -2 b) 1

5*. What is the left hand limit of the function $f(x)=\dfrac{7|x|-xe^x\sin\left(\frac{\pi|x|}{2x}\right)}{x^2+3x}$?

6*. Find the constant c such that $\lim\limits_{x \to c} \dfrac{x^2-5x+6}{x-c}$ exists. **Answer**: $c=-1, 6$

7. Show that $\sin x = x$ has exactly one solution.

8. Evaluate the following limits

a) $\lim\limits_{x\to 0}\dfrac{\tan ax}{\tan bx}$ b) $\lim\limits_{x\to 0}\dfrac{\cot 2x}{\cot 4x}$ c) $\lim\limits_{x\to 0}\dfrac{1}{x^4\csc^4 3x}$ d) $\lim\limits_{x\to 0}\dfrac{\tan 5x}{\cot 3x}$

e) $\lim\limits_{x\to\infty}\left(\dfrac{x}{x-1}\right)^x$ f) $\lim\limits_{x\to\infty}\left(\dfrac{3x+4}{3x+6}\right)^{3x}$ g) $\lim\limits_{x\to\infty}\left(\dfrac{x+2}{x+1}\right)^x$ h) $\lim\limits_{x\to 0^+}(1+2x)^{\frac{1}{x}}$

Answer : a) $\frac{a}{b}$ b) 2 c) 81 d) 15 e) e f) $\frac{1}{e^2}$ g) e h) e^2

9*. Given $\psi(3x)=\dfrac{x}{x^2+1}$. Evaluate $\lim\limits_{x\to 1}\dfrac{\psi(x)-\psi(1)}{x-1}$. **Answer** : $\frac{6}{25}$

10. Suppose $1-x^2\le f(x)-e^x\le \cos x, \forall x\in(-\pi/2,\pi/2)$. Evaluate $\lim\limits_{x\to 0} f(x)$.

11. Find a and k so that the functions will be continuous at $x=0$.

a) $f(x)=\begin{cases}\dfrac{\sin([1-k^2]x)}{x}, & x\ne 0\\ 2k-2, & x=0\end{cases}$ b) $f(x)=\begin{cases}x+1, & x<2\\ x^2+kx+1, & x\le 2\end{cases}$

c*) $g(x)=\begin{cases}\frac{k^2}{2}\sin(\frac{\pi|x|}{2x}), & x<0\\ 3k+\frac{9}{2}e^{kx}, & x\ge 0\end{cases}$ **Answer** : a) k = 1,-3 b) $k=-1$

12. Find the constant c such that

a) $\lim\limits_{x\to\infty}\left(\dfrac{x+c}{x-c}\right)^x=9$ b) $\lim\limits_{x\to\infty}\left(\dfrac{x-c}{x+c}\right)^x=\dfrac{1}{4}$ c) $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+c}\right)^x=e^3$

13. Show that $\cos x = x$ has a solution on $[0,\pi/2]$.

14. Evaluate $\lim\limits_{x\to\infty}\dfrac{6x^2+x\sin(\sqrt{x})}{x^2+12}$ using Squeezing Rule.

15. Show that $f(x)=x^4+5x^3+5x-1$ has at least two real solutions in $[-6,2]$.

16. If $\dfrac{\tan(12x)}{\tan(2x)}\le 2e^{f(x)}\le e^{-2x}+5$ in an interval containing 0, find $\lim\limits_{x\to 0} f(x)$.

17. Suppose $|x|+2\le f(x)+x^2\le\dfrac{\sin 2x}{x}$. Evaluate $\lim\limits_{x\to 0} f(x)$. **Answer** : 2

18. Prove that every polynomial of odd degree has at least one real root on R.

CHAPTER-5

DERIVATIVE AND ITS APPLICATIONS

Introduction: Why Calculus?

Every subject area or field of specialization arises from the need to solve a problem and from a need to invent new invention while exploring our physical world. Likewise, many branches of Mathematics were developed due to a physical problem that the ancient people were facing. Particularly, ***Differential and Integral Calculus*** were developed due to the following physical problems encountered during the seventeenth century.

1. **Tangent line Problems:** How to find the slope of a tangent line to an arbitrary curve?
2. **Problems of Motion:** How to find the instantaneous velocity and acceleration of a moving body in a very small change of time?
3. **Modeling (Optimization Problems):** How to find maximum and minimum values of a model developed for physical problem?
4. **Area Problems:** How to find area of an irregular shaped region?

In real situation,to any one who has no knowledge of Calculus, the answer to these problems is really difficult. But for Science, Enginnering,Technology and other Educational experts familiar to calculus, the answer to these problems are like the answer to the question '*Good Morning ever?'(Well this is what Psychologists say Minichis-Minichis).* '*Oh!* don't worry to know the meaning of this proverb but guess the meaning by yourself from the context'.Therefore, our next obectives focus on answering these basic and commonly encountred problems by dealing both ***Differential and Integral Calculus.***

5.1 Definition and Examples

Definition: The derivative of a function f at a number a in the domain of f denoted by $f'(a)$ is defined as

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ or } f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$ provided that this limit exists. If this limit exists, we say that the function f has a derivative at $x=a$ and f is said to be differential at $x=a$.

But this limit exists if $\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$ and $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$ both exists.

Here, $\lim_{x\to a^-}\frac{f(x)-f(a)}{x-a}$ is called left hand derivative and $\lim_{x\to a^+}\frac{f(x)-f(a)}{x-a}$ is called right hand derivative. Then a function is differentiable at $x=a$ only if the left hand and right hand derivatives are equal.

Examples:

1. Using the definition, find the derivatives of

a) $f(x)=x^2+6$ *b)* $f(x)=\sqrt{2x+1}$ *c)* $f(x)=\frac{1}{x}$

d) $f(x)=x^2-5x$ at $a=5$ *e)* $f(x)=x^{\frac{1}{3}}$ at $a=0$

Solution:

a) $f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{x^2-a^2}{x-a}=\lim_{x\to a}\frac{(x-a)(x+a)}{x-a}=\lim_{x\to a}(x+a)=2a$

b) $f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}=\lim_{x\to a}\frac{\sqrt{2x+1}-\sqrt{2a+1}}{x-a}$

$$=\lim_{x\to a}\frac{(\sqrt{2x+1}-\sqrt{2a+1})}{x-a}\left(\frac{\sqrt{2x+1}+\sqrt{2a+1}}{\sqrt{2x+1}+\sqrt{2a+1}}\right)$$

$$=\lim_{x\to a}\frac{2(x-a)}{(x-a)\sqrt{2x+1}+\sqrt{2a+1}}=\lim_{x\to a}\frac{2}{\sqrt{2x+1}+\sqrt{2a+1}}=\frac{1}{\sqrt{2a+1}}$$

c) $h'(a)=\lim_{x\to a}\frac{h(x)-h(a)}{x-a}=\lim_{x\to a}\frac{\frac{1}{x}-\frac{1}{a}}{x-a}=\lim_{x\to a}\frac{a-x}{ax(x-a)}=\lim_{x\to a}\frac{-1}{ax}=-\frac{1}{a^2}$

2. Given $f'(1)=8$. Evaluate the following g expressions.

$a)\ \lim_{x\to1}\frac{f(x)-f(1)}{\sqrt{x}-1}$ $\qquad b)\ \lim_{x\to1}\frac{f(x)-f(1)}{x^4-1}$

Solution:

$$a)\ f'(1)=\lim_{x\to1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to1}\frac{f(x)-f(1)}{(\sqrt{x}-1)(\sqrt{x}+1)}$$

$$\Rightarrow\lim_{x\to1}\frac{f(x)-f(1)}{\sqrt{x}-1}=f'(1).\lim_{x\to1}(\sqrt{x}+1)=8(\sqrt{1}+1)=8(2)=16$$

$$b)\ \lim_{x\to1}\frac{f(x)-f(1)}{x^4-1}=\lim_{x\to1}\frac{f(x)-f(1)}{(x-1)(x+1)(x^2+1)}$$

$$=\lim_{x\to1}\frac{f(x)-f(1)}{x-1}.\lim_{x\to1}\frac{1}{(x+1)(x^2+1)}$$

$$=f'(1).\lim_{x\to1}\frac{1}{(x+1)(x^2+1)}=\frac{8}{4}=2$$

3. Suppose f is a differentiable function with the property that

$f(x+y)=f(x)+f(y)+20xy,\ \forall x,y\in R$ and $\lim_{h\to0}\frac{f(h)}{h}=-15$.

Find $f(0), f'(x)$ and $f'(2)$.

Solution: From the given property, $f(0)=f(0+0)=f(0)+f(0)\Rightarrow f(0)=0$

Besides, using the definition of derivative, we have

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f(x)+f(h)+20xh-f(x)}{h}$$

$$=\lim_{h\to0}\frac{f(h)+20xh}{h}=20x+\lim_{h\to0}\frac{f(h)}{h}=20x-15$$

Hence, $f'(2)=20(2)-15=40-15=25$.

4. If f is differentiable and $f(a)=0$, $f'(a)=21$, evaluate $\lim_{h\to0}\frac{f(a+h)}{7h}$.

Solution:

$$\lim_{h\to0}\frac{f(a+h)}{7h}=\frac{1}{7}\lim_{h\to0}\frac{f(a+h)-0}{h}=\frac{1}{7}\lim_{h\to0}\frac{f(a+h)-f(a)}{h}=\frac{1}{7}f'(a)=3$$

5.2 Differentiability and Continuity

A) **Differentiability implies continuity:** If a function is differentiable at $x=a$, it is continuous at $x=a$. That means every differentiable function is continuous. The converse of the theorem is not true. That means every continuous function need not be differentiable.

B) **Criteria of Differentiability for piecewise functions:** Consider piece wise function of the form $f(x)=\begin{cases} g(x), x\le a \\ h(x), x>a \end{cases}$

i) The function f is said to be differentiable at $x=a$ only if $g'(a)=h'(a)$. But if $g'(a)\neq h'(a)$, then we say that f is not differentiable at $x=a$.

ii) The function f is said to be continuous at $x=a$ only if $g(a)=h(a)$.

Generalization: Notice! Notice! Notice!

For any function f to be differentiable at $x=a$, the left hand derivative $f'(a)$ for $x\le a$ and the right hand derivative $f'(a)$ for $x>a$ must be equal.

That means $f'(a)$ exists if and only if $\underset{\text{for } x\le a}{f'(a)}=\underset{\text{for } x>a}{f'(a)}$.

Examples:

1. Show that f is continuous but not differentiable at the given point.

a) $f(x)=|x-7|$ at x = 7 b) $f(x)=\begin{cases} x^3+3, x\le 1 \\ 5x^2-1, x>1 \end{cases}$ at x = 1

Solution:

a) Using the definition of absolute value, $f(x)=|x-7|=\begin{cases} x-7, x\ge 7 \\ 7-x, x<7 \end{cases}$

Here, $\lim_{x\to 7^-} f(x)=\lim_{x\to 7^-}(7-x)=0$, $\lim_{x\to 7^+} f(x)=\lim_{x\to 7^+}(x-7)=0 \Rightarrow \lim_{x\to 7} f(x)=0$

Besides, $f(7)=0 \Rightarrow \lim_{x\to 7} f(x)=f(7)$. Hence, it is continuous at $x=7$.

By the above criteria, a function $f(x)=\begin{cases} g(x), x\le a \\ h(x), x>a \end{cases}$ is differentiable at $x=a$ only if $g'(a)=h'(a)$.

However, $f'(x)=\begin{cases}1, x\geq 7\\ -1, x<7\end{cases}$ with $1\neq -1$.

Hence, $f(x)=|x-7|$ is not differentiable at $x=7$.

b) Here, $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^-}(x^3+3)=4, \lim_{x\to 1^+} f(x)=\lim_{x\to 1^+}(5x^2-1)=4 \Rightarrow \lim_{x\to 1} f(x)=4$

Besides, $f(1)=4 \Rightarrow \lim_{x\to 1} f(x)=f(1)=4$. Hence, it is continuous at $x=1$.

Now, we need to check the differentiability of f at $x=1$.

Here, $f'(x)=\begin{cases}g(x)=3x^2, x\leq 1\\ h(x)=10x, x>1\end{cases} \Rightarrow f'(1)=\begin{cases}3, x\leq 1\\ 10, x>1\end{cases}$

However, $f'(1)=3$ for $x\leq 1$ and $f'(1)=10$ for $x>1$ are different.

Hence, f is not differentiable at $x=1$.

2. Find the constants a and b if f is differentiable at the given point.

a) $f(x)=\begin{cases}ax^2-5x+4, x\leq 2\\ 7x+b\ , x>2\end{cases}$ at $x=2$ b) $f(x)=\begin{cases}2x^3+1,\ x\leq 1\\ 3ax+b, x>1\end{cases}$ at $x=1$

Solution:

a) For $f'(x)=\begin{cases}2ax-5,\ x\leq 2\\ 7, x>2\end{cases}$ to be differentiable at $x=2$, the right and left derivative must be equal at $x=2$. That is $4a-5=7 \Rightarrow 4a=12 \Rightarrow a=3$.

Besides, every differentiable function is continuous.

So, $\lim_{x\to 2^+} f(x)=\lim_{x\to 2^-} f(x)=f(2)$.

Hence, using $a=3$, we have $4a-10+4=14+b \Rightarrow 6=14+b \Rightarrow b=-8$

b) For $f'(x)=\begin{cases}6x^2, x\leq 1\\ 3a\ , x>1\end{cases}$ to be differentiable at $x=1$, the right and the left derivative must be equal at $x=1$. That is $3a=6 \Rightarrow a=2$.

Besides, $\lim_{x\to 1^+} f(x)=\lim_{x\to 1^-} f(x)=f(1) \Rightarrow 3a+b=3 \Rightarrow 6+b=3 \Rightarrow b=-3$.

5.3 Derivatives of Different Functions

I) Derivatives of Trigonometric functions:

The derivative of sine:

For $g(x)=\sin x$, by definition

$$g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}=\lim_{h\to 0}\frac{\sin(x+h)-\sin x)}{h}=\lim_{h\to 0}\frac{\sin x\cosh+\cos x\sinh-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x[\cosh-1]}{h}+\lim_{h\to 0}\frac{\cos x\sinh}{h}=\sin x\lim_{h\to 0}\frac{\cosh-1}{h}+\cos x\lim_{h\to 0}\frac{\sinh}{h}$$

$$=\sin x \qquad \therefore\ \lim_{h\to 0}\frac{\cosh-1}{h}=0,\quad \lim_{h\to 0}\frac{\sinh}{h}=1$$

Similarly, the derivatives of the other trigonometric functions can be derived.

Derivative of cosine: For $f(x)=\cos x$. Then, $f'(x)=-\sin x$

Derivative of tangent: For $f(x)=\tan x$. Then, $f'(x)=\sec^2 x$.

Derivative of secant: For $f(x)=\sec x$. Then, $f'(x)=\sec x\tan x$.

Derivative of cotangent: For $f(x)=\cot x$. Then, $f'(x)=-\csc^2 x$.

Derivative of cosecant: For $f(x)=\csc x$. Then, $f'(x)=-\csc x\cot x$.

II) Derivatives of Exponential Functions:

Suppose $f(x)=a^x$, $a>0$, $a\neq 1$. Then,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{a^x(a^h-1)}{h}=a^x\lim_{h\to 0}\frac{a^h-1}{h}=a^x\ln a$$

Examples: If $f(x)=2^x$, then $f'(x)=2^x\ln 2$.

III) Derivatives of Logarithmic functions

Derivatives of Natural Logarithm:

Suppose $f(x)=\ln x$, $x>0$. Then,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}=\lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h},\quad \because \ln(x+h)-\ln x=\ln\frac{x+h}{x}$$

$$=\lim_{h\to 0}\frac{\ln\left(\frac{x+h}{x}\right)}{h}=\lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right)=\lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}=\ln\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{\frac{1}{h}}=\ln e^{\frac{1}{x}}=\frac{1}{x}$$

Derivatives of Logarithms of any base:

Suppose $f(x) = \log_a x, a > 0, a \neq 1, x > 0$.

Then, $f(x) = \log_a x = \frac{\ln x}{\ln a} \Rightarrow f'(x) = (\log_a x)' = (\frac{\ln x}{\ln a})' = \frac{1}{\ln a}(\ln x)' = \frac{1}{x\ln a}$

Example: $f(x) = \log_3 x \Rightarrow f'(x) = \frac{1}{x\ln 3}$.

5.4 Derivatives of Combinations of Functions

Here, we are going to see the basic rules to find the derivatives of combinations like sum, difference, constant multiple, product and quotient of functions.

Rules of Derivatives: Suppose f and g are differentiable functions and k is any constant. Then,

The power rule: $f(x) = x^k$, $f'(x) = kx^{k-1}$.

Constant multiple rule: $(kf)'(x) = kf'(x)$

Sum Rule: $(f+g)'(x) = f'(x) + g'(x)$.

Difference rule: $(f-g)'(x) = f'(x) - g'(x)$.

Product rule: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Quotient rule: $(\frac{f}{g})'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$ and $(\frac{1}{f})'(x) = \frac{-f'(x)}{[f(x)]^2}$

Constant rule: The Derivative of a constant function is zero.

Examples: How to use these rules:

a) Let $f(x) = 5x^6$. Then, by power rule, $f'(x) = (5x^6)' = 5(x^6)' = 30x^5$.

b) Let $f(x) = 2x^4 + \sin x$. By sum rule, $f'(x) = (2x^4)' + (\sin x)' = 8x^3 + \cos x$.

c) Product: $f(x) = x^3 \cos x \Rightarrow f'(x) = (x^3)'\cos x + x^3(\cos x)' = 3x^2\cos x - x^3 \sin x$.

d) Quotient rule: $f(x) = \frac{x^4}{x^4+5} \Rightarrow f'(x) = \frac{(x^4)'(x^4+5) - x^4(x^4+5)'}{(x^4+5)^2}$

$$= \frac{4x^3(x^4+5) - x^4(4x^3)}{(x^4+5)^2} = \frac{20x^3}{(x^4+5)^2}$$

5.5 The Chain Rule: Derivative of Composite Function

The chain rule is the basic rule which is used to find the derivatives of compositions of functions. It is stated as a theorem below for two functions.

Theorem: Suppose g is differentiable at x and f is differentiable at $g(x)$. Then, the derivative of $f \circ g$ is given by $(f \circ g)'(x) = f'(g(x))g'(x)$.

Remarks: Important Hints and Examples in Chain Rule:

Note that the main problem in differentiating composition of functions using chain rule is that students are not capable of identifying $f(x)$ and $g(x)$ from the given formula. Of course there is no strict rule to follow. But remembering fundamental or basic functions will help to do so. Now, in order to apply chain rule effectively, let's see some useful hints based on the forms of the functions.

Derivative of the form: $f(x)=[g(x)]^n$, we get $f'(x)=n.[g(x)]^{n-1}.g'(x)$

Example: Find the derivative of $f(x)=(x^3+1)^6$.

By chain rule, $f'(x)=6(x^3+1)^5(x^3+1)'=6(x^3+1)^5(3x^2)=18x^2(x^3+1)^4$

Derivative of: $f(x)=\sqrt{g(x)}$, we have $f'(x)=\dfrac{g'(x)}{2\sqrt{g(x)}}$.

Example: For $f(x)=\sqrt{x^4+1}$, we get $f'(x)=\dfrac{(x^4+1)'}{2\sqrt{x^4+1}}=\dfrac{2x^3}{\sqrt{x^4+1}}$.

Derivative of the form: $f(x)=e^{g(x)}$, we have $f'(x)=e^{g(x)}.g'(x)$

Examples:

a) For $f(x)=e^{x^3}$, we get $f'(x)=e^{x^3}.(x^3)'=3x^2e^{x^3}$

b) For $f(x)=e^{\sin x}$, we have $f'(x)=e^{\sin x}(\sin x)'=\cos x e^{\sin x}$

Derivative of the form: $f(x)=a^{g(x)}$, we have $f'(x)=a^{g(x)}.g'(x)\ln a$

Example: For $f(x)=2^{x^3}$, $f'(x)=2^{x^3}.(x^3)'\ln 2=3x^2 2^{x^3}\ln 2$.

Derivative of the form: $f(x)=\ln(g(x))$, we have $f'(x)=\dfrac{g'(x)}{g(x)}$

Example: For $f(x)=\ln(5x^2+4)$, we get $f'(x)=\dfrac{(5x^2+4)'}{5x^2+4}=\dfrac{10x}{5x^2+4}$

Derivative of $f(x)=\cos(g(x))$: We have $f'(x)=-\sin(g(x)).g'(x)$

Example: For $f(x)=\cos(2x^3)$, $f'(x)=-\sin(2x^3).(2x^3)'=-6x^2\sin(2x^3)$

Derivative of the form: $f(x)=\sin(g(x))$, we get $f'(x)=\cos(g(x)).g'(x)$

Example: For $f(x)=\sin(5x^2)$, $f'(x)=\cos(5x^2).(5x^2)'=10x\cos(5x^2)$.

Derivative of $f(x)=\tan(g(x))$: We have $f'(x)=\sec^2(g(x)).g'(x)$

Example: For $f(x)=\tan(x^4)$, $f'(x)=\sec^2(x^4)(x^4)'=4x^3\sec^2(x^4)$

Compound chain rule: The chain rule can be extended to any number of finite compositions of functions. The rule is known as Compound chain rule.

For three functions, it becomes $(f\circ g\circ h)'(x)=f'(g\circ h(x))g'(h(x))h'(x)$.

Examples:

1. Use compound chain rule to find the following derivatives.

$a)\ f(x)=e^{\cos(x^2)}$ $\quad c)\ ^*h(x)=\cos(e^{x^3})$ $\quad c)\ f(x)=2\tan^2(x^7)$

$d)\ f(x)=\cos^6(x^2)$ $\quad e)\ f(x)=e^{\sin 3x}$ $\quad f)\ f(x)=\ln(\cos e^x)$

$g)\ f(x)=\sqrt{\sin 4x+e^x}$ $\quad h)\ f(x)=\ln(\sin(e^{2x}))$ $\quad i)\ f(x)=\ln(\sin^2(e^{x^2}))$

Solution: Apply compound chain rule.

$a)\ f(x)=e^{\cos x^2}\Rightarrow f'(x)=e^{\cos x^2}(\cos x^2)'(x^2)'=-2xe^{\cos x^2}\sin x^2$

$b)\ h'(x)=-\sin(e^{x^3})(e^{x^3})'=-e^{x^3}\sin(e^{x^3})(x^3)'=-3x^2e^{x^3}\sin(e^{x^3})$

$c)\ f'(x)=4\tan(x^7)\sec^2(x^7)(x^7)'=28x^6\tan(x^7)\sec^2(x^7)$

$d)\ h'(x)=6\cos^5(x^2)\sin(x^2)(x^2)'=12x\cos^5(x^2)\sin(x^2)$

$e)\ f'(x)=e^{\sin 3x}(\sin 3x)'=e^{\sin 3x}\cos 3x(3x)'=3e^{\sin 3x}\cos 3x$

$$f)\ f'(x)=\frac{(\sin 4x+e^x)'}{2\sqrt{\sin 4x+e^x}}=\frac{4\cos 4x+e^x}{2\sqrt{\sin 4x+e^x}}$$

$$g)\ f'(x)=\frac{1}{\sin(e^{2x})}\left(\sin(e^{2x})\right)=\frac{\cos(e^{2x})}{\sin(e^{2x})}(e^{2x})'=2e^{2x}\cot(e^{2x})$$

$$h)\ f'(x)=\frac{1}{\sin^2(e^{x^2})}\left(\sin^2(e^{x^2})\right)=\frac{2\sin(e^{x^2})}{\sin^2(e^{x^2})}(e^{x^2})'=4xe^{x^2}\csc(e^{x^2})$$

2. Suppose $g(x)=\frac{f(x)}{x-1}-2f(x)$. If $f(3)=-10, f'(3)=7$, then find $g'(3)$.

Solution: We can apply quotient and difference rules as follow:

That is , $g'(x)=\frac{f'(x)(x-1)-f(x)}{(x-1)^2}-2f'(x)$.

Therefore, $g'(3)=\frac{2f'(3)-f(3)}{4}-2f'(3)=\frac{24}{4}-14=6-14=-8$.

3. Suppose $g(x)=(4-x^2)f(x^3+1)$ where $f'(x)=\sqrt{e^x+3}$. If $f(0)=3$, then find $g'(-1)$.

Solution: We can apply product and chain rules as follow:

That is $g'(x)=-2xf(x^3+1)+3x^2(4-x^2)f'(x^3+1)$.

Therefore, $g'(-1)=2f(0)+9f'(0)=2(3)+9(2)=6+18=24$.

4. Given $f(2)=3, f'(2)=4, g(0)=3, g'(0)=5$. If $G(x)=f(x^3+x)g(1-x^2)$, then, find $G'(1)$.

Solution: Apply product and chain rules on $G(x)=f(x^3+x)g(1-x^2)$.

That is $G'(x)=(3x^2+1)f'(x^3+x)g(1-x^2)-2xf(x^3+x)g'(1-x^2)$

Hence, $G'(1)=4f'(2)g(0)-2f(2)g'(0)=4(4)(3)-2(3)(5)=48-30=18$

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Chain rule for Derivatives of the form $y=[f(x)]^{g(x)}$:

The derivatives of such function is obtained by converting into logarithmic function and then applying logarithmic differentiation as follow.

$$y=[f(x)]^{g(x)} \Rightarrow \ln y=\ln[f(x)]^{g(x)}=g(x)\ln[f(x)]$$

$$\Rightarrow \frac{d}{dx}[\ln y]=\frac{d}{dx}(g(x)\ln[f(x)]) \Rightarrow \frac{1}{y}\frac{dy}{dx}=g'(x)\ln[f(x)]+g(x)\frac{f'(x)}{f(x)}$$

$$\Rightarrow \frac{dy}{dx}=[f(x)]^{g(x)}\left[g'(x)\ln[f(x)]+g(x)\frac{f'(x)}{f(x)}\right]$$

Therefore, if $y=[f(x)]^{g(x)}$, $y'=\frac{dy}{dx}=[f(x)]^{g(x)}\left[g'(x)\ln[f(x)]+g(x)\frac{f'(x)}{f(x)}\right]$.

Examples: Find the derivatives of the following functions.

$a)\ y=(x^2+2)^{\ln x}$ $\quad b)\ y=x^{\ln x}$ $\quad c\ y=(x^3+2x)^{x^5}$ $\quad d)\ f(x)=x^{\sin x}$

$e)\ y=x^x$ $\quad f)\ y=x^{\sec x}$ $\quad g)\ y=x^{x^2}$ $\quad h)\ y=\sqrt{x}^{\sqrt{x}}$

Solution: Identify $f(x)$ and $g(x)$ to use the derivative formula of $y=[f(x)]^{g(x)}$

$$a)\ y'=(x^2+2)^{\ln x}\left[(\ln x)'\ln(x^2+2)+\ln x.\frac{(x^2+2)'}{x^2+2}\right]$$

$$=(x^2+2)^{\ln x}\left[\frac{\ln(x^2+2)}{x}+\frac{2x\ln x}{x^2+2}\right]$$

$$b)\ y'=x^{\ln x}\left[\frac{\ln x}{x}+\frac{\ln x}{x}\right]=x^{\ln x}\left[(\ln x)'\ln x+\ln x.(\frac{1}{x})\right]=2x^{\ln x}\frac{\ln x}{x}$$

$$c)\ y'=(x^3+2x)^{x^5}\left[\ln(x^3+2x)(x^5)'+(x^5)\frac{(x^3+2x)'}{x^3+2x}\right]$$

$$=(x^3+2x)^{x^5}\left[5x^4\ln(x^3+2x)+\frac{3x^6+2x^4}{x^2+2}\right]$$

$$d)\ y'=x^{\sin x}\left[(\sin x)'\ln x+\sin x.(\frac{1}{x})\right]=x^{\sin x}\left[\cos x\ln x+\frac{\sin x}{x}\right]$$

$e)\ y'=x^x[\ln x+1]$ $\quad f)\ y'=x^{\sec x}\left[\sec x\tan x\ln x+\frac{\sec x}{x}\right]$ $\quad g)\ y'=x^{x^2}[2x\ln x+x]$

5.6 Derivatives of Inverses of Transcendental Functions

5.6.1 Definition and Examples of Transcendental Functions

Algebraic and Transcendental Function: A function f is called an *algebraic function* provided that $y = f(x)$ satisfies some polynomial equation in y of the form $a_n(x)y^n + a_{n-1}(x)y^{n-1} + ... + a_1(x)y + a_0(x) = 0$ where the coefficients $a_0(x), a_1(x), a_2(x),..., a_n(x)$ of y^n are polynomials in x. In short, an algebraic function of x is a function that can be expressed as a finite sum, difference, product, quotient, and radicals or n^{th} – roots involving polynomials of x.

Any function that is not algebraic is said to be *transcendental* function.

Examples

a) $y = \sqrt{x+3}$ is algebraic because it satisfies the equation $y^2 - x - 3 = 0$.

b) $y = \dfrac{x^2 - 5x + 7}{x^3 + 1}$ is algebraic because it is a quotient of two polynomials and satisfies $(x^3 + 1)y - x^2 + 5x - 7 = 0$.

c) $y = \dfrac{2x^2}{\sqrt[3]{x^5 + 4}}$ is algebraic because it satisfies $(x^5 + 4)y^3 - 8x^6 = 0$.

d) Is $y = \sqrt{x + \sqrt{x}}$ algebraic? If it is algebraic, find the equation it satisfies. Here,

$$y = \sqrt{x + \sqrt{x}} \Rightarrow y^2 = x + \sqrt{x} \Rightarrow y^2 - x = \sqrt{x}$$
$$\Rightarrow (y^2 - x)^2 = x \Rightarrow y^4 - 2xy^2 + x^2 - x = 0$$

This means the function satisfies the polynomial equation $y^4 - 2xy^2 + x^2 - x = 0$ in y with polynomial coefficients in x. Hence, it is algebraic function.

Note: In general, for any polynomial $p(x)$, $y = \sqrt[n]{p(x)}$ is algebraic.

e) The functions $y = 2^x, y = \ln x, y = \tan x$ are not algebraic. (Do you see why?)

General Classifications:

I) Class of Algebraic Functions:

a) Polynomial functions *b) Rational functions*

c) Radical or n^{th} – roots *of polynomial and Rational functions*

All these functions and any combinations of them are algebraic functions.

II) Class of Transcendental Functions:

a) Exponential functions

b) Logarithmic functions

c) Trigonometric functions and their Inverses

d) Hyperbolic Functions and their Inverses

All these functions and any combinations of them are transcendental.

5.6.2 Test for Existence of Inverses

We know that every function need not have inverse. So, before trying to find the derivative of inverses of different functions, the main task is to develop techniques how to identify whether the given function has inverse or not. There are a number of techniques to check existence of inverse: Horizontal line Test or Geometric Test, Bi-jectivity Test and Derivative Test. But as an Applied Course, in this material we will discuss the Derivative Test.

Derivative Test:

A non-constant differentiable function f will have an inverse provided that $f'(x) \geq 0$ **OR** $f'(x) \leq 0$ on its entire or whole domain.

Notice about the test:

When we say on the whole domain, it means that only one of the inequality must be true for all domain members.

But if the domain set is subdivided or separated into two subsets so that $f'(x) \geq 0$ on some part and $f'(x) \leq 0$ on the other part of the domain, then f has no inverse in its entire domain.

Examples:

1. Which of the following function has an inverse in its entire domain?

$a)\ f(x)=x^5+2x^3+3x+4 \quad b)\ g(x)=3-x^9 \quad c)\ f(x)=4+x^6$

$d)\ f(x)=x^{11}+x^9+4x+6 \quad e)\ f(x)=xe^{-x} \quad f)\ f(x)=e^{3x}+2x$

Solution: Apply the Test for existence of inverse.

a) Since f is a polynomial, its domain is the set of all real numbers.

Besides, as we see from its derivative $f'(x)=5x^4+6x^2+3$, all the powers are even and its constant term is positive. So, $f'(x)>0$ on the whole domain. That is $f'(x)=5x^4+6x^2+3>0, \forall x\in R$. Hence, f has an inverse.

b) Here, $g(x)=3-x^9 \Rightarrow g'(x)=-9x^8\le 0, \forall x\in R$. Hence, by first derivative test, $g(x)$ is decreasing on R. Hence, g has an inverse.

c) Since f is a polynomial, its domain is the set of all real numbers.

But $f'(x)=6x^5$ satisfies $f'(x)\ge 0$ for $x\ge 0$ and $f'(x)\le 0$ for $x<0$.

This means $f'(x)\ge 0$ for some part of the domain and $f'(x)\le 0$ for the other part. Hence, f has no inverse on the whole of its domain.

d) Here, $f'(x)=11x^{10}+9x^8+4$. Since all the powers are even and all the coefficients as well as the constant term are positive $f'(x)>0$.

Hence, f is increasing on the whole domain R. Therefore, it has an inverse.

e) Here, $f'(x)>0 \Rightarrow e^{-x}(1-x)>0 \Rightarrow x<1$ and $f'(x)<0 \Rightarrow e^{-x}(1-x)<0 \Rightarrow x>1$

This means f is not increasing or decreasing on the whole of its domain.

Therefore, f has no inverse on the whole of its domain.

2. Find the inverse by restricting the domain where the inverse is valid.

$a)\ f(x)=\sqrt{x^2+6x} \quad b)\ f(x)=x^2-2x \quad c)\ f(x)=2x^4+6$

Solution: Notice that the interval where the inverse of a function is valid is the interval where the function is either increasing or decreasing. So, using the ***First Derivative Test for Monotone***, the inverse of the function in part (a), is valid on the interval $\{x: x\ge 0\}$ or $\{x: x\le -6\}$, that of in part (b) is valid on $\{x: x\ge 1\}$ or $\{x: x\le 1\}$, that of in part (c) is valid on $\{x: x\ge 0\}$ or $\{x: x\le 0\}$.

a) To find the inverse, first, interchange x and y in $y = f(x) = \sqrt{x^2 + 6x}$ and then, solve for y in terms of x.

That is $x = \sqrt{y^2 + 6y} \Rightarrow y^2 + 6y = x^2 \Rightarrow y^2 + 6y - x^2 = 0$

$$\Rightarrow y = \frac{-6 \pm \sqrt{36 + 4x^2}}{2} = -3 \pm \sqrt{9 + x^2}$$

Hence, on $\{x : x \geq 0\}$, $f^{-1}(x) = -3 + \sqrt{9 + x^2}$ (Why?) and on $\{x : x \leq -6\}$, $f^{-1}(x) = -3 - \sqrt{9 + x^2}$ (Why?)

b) First, interchange x and y in the formula $y = f(x) = x^2 - 2x$. That is $x = y^2 - 2y$. Then, solve for y. That is

$x = y^2 - 2y \Rightarrow y^2 - 2y = x \Rightarrow y^2 - 2y - x = 0$

$$\Rightarrow y = \frac{2 \pm \sqrt{4 + 4x}}{2} = 1 \pm \sqrt{1 + x}$$

So, on $\{x : x \geq 1\}$, $f^{-1}(x) = 1 + \sqrt{1 + x}$, on $\{x : x \leq 1\}$, $f^{-1}(x) = 1 - \sqrt{1 + x}$

(Do you see Why?)

5.6.3 Derivative of Inverse Functions

Derivative of Inverse: Suppose f has an inverse and is continuous on an open interval I containing a such that $f'(a)$ exists and $f'(a) \neq 0$.

If $f(a) = c$, then $(f^{-1})'(c)$ also exists and $(f^{-1})'(c) = \dfrac{1}{f'(a)}$.

In general, if $y = f(x)$ and $f'(x) \neq 0$, then $(f^{-1})'(x) = \dfrac{1}{f'(x)} = \dfrac{1}{dy/dx} = \dfrac{dx}{dy}$.

Summary of Steps to find $(f^{-1})'(c)$:

First: Find a domain element a using the relation $f(a) = c$

Second: Find $f'(x)$ using the given expression and evaluate $f'(a)$

Third: Apply the formula $(f^{-1})'(c) = \dfrac{1}{f'(a)}$

Examples:

1. Find $(f^{-1})'(c)$ where

a) $f(x)=x^3+1;c=28$

b) $f(x)=x^5+3x^3+4x+2;c=2$

c) $f(x)=x+\sqrt{x},c=12$

d) $f(x)=\ln(3-\sqrt{x});c=0$

e) $f(x)=e^x-3e^{-x};c=2$

f) $f(x)=2+\ln x^3;c=5$

Solution:

a) To find $(f^{-1})'(c)$ using the above theorem, first find a such that $f(a)=c$.

But, $f(a)=c \Rightarrow f(a)=28 \Rightarrow 1+a^3=28 \Rightarrow a^3=27 \Rightarrow a=3$.

Thus, $(f^{-1})'(c)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(28)=\frac{1}{f'(3)}=\frac{1}{27}$.

b) To find $(f^{-1})'(c)$ using the above theorem, first find a such that $f(a)=c$.

$f(a)=c \Rightarrow a^5+3a^3+4a+2=2 \Rightarrow a^5+3a^3+4a=0$

$\Rightarrow a(a^4+3a^2+4)=0 \Rightarrow a=0$

Besides, $f'(x)=5x^4+9x^2+4 \Rightarrow f'(0)=4$.

Thus, $(f^{-1})'(c)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(2)=\frac{1}{f'(0)}=\frac{1}{4}$.

c) $f(x)=x+\sqrt{x} \Rightarrow a+\sqrt{a}=12 \Rightarrow \sqrt{a}=12-a$

$\Rightarrow (\sqrt{a})^2=(12-a)^2 \Rightarrow a^2-25a+144=0$.

$\Rightarrow (a-9)(a-16)=0 \Rightarrow a=9, a=16$

(Do you see why $a=16$ is not a valid solution? Explain!)

Hence, $(f^{-1})'(c)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(12)=\frac{1}{f'(9)}=\frac{6}{7}$.

d) $f(a)=c \Rightarrow \ln(3-\sqrt{a})=0 \Rightarrow 3-\sqrt{a}=e^0 \Rightarrow 3-\sqrt{a}=1$

$\Rightarrow \sqrt{a}=2 \Rightarrow (\sqrt{a})^2=4 \Rightarrow a=4$

Besides, $f'(x)=\frac{1}{3-\sqrt{x}}.(-\frac{1}{2\sqrt{x}})=-\frac{1}{6\sqrt{x}-2x} \Rightarrow f'(4)=-\frac{1}{4}$.

Hence, $(f^{-1})'(c)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(0)=\frac{1}{f'(4)}=-4$.

e) $f(a)=c \Rightarrow e^a-3e^{-a}=2 \Rightarrow e^{2a}-2e^a-3=0$. Using substitution $e^a=t$,

we have $e^{2a}-2e^a-3=0 \Rightarrow t^2-2t-3=0 \Rightarrow (t-3)(t+1)=0 \Rightarrow t=3,-1$

Now $e^a=t \Rightarrow e^a=3 \Rightarrow a=\ln 3$ but $e^a=-1$ is not possible.

Besides, $f(x)=e^x-3e^{-x} \Rightarrow f'(x)=e^x+3e^{-x} \Rightarrow f'(\ln 3)=3+1=4$

Hence, $(f^{-1})'(c)=\dfrac{1}{f'(a)} \Rightarrow (f^{-1})'(2)=\dfrac{1}{f'(\ln 3)}=\dfrac{1}{4}$.

f) First find a that satisfies the condition $f(a)=c$.

But, $f(a)=5 \Rightarrow 2+\ln a^3=5 \Rightarrow \ln a^3=3 \Rightarrow a^3=e^3 \Rightarrow a=e$.

Thus, $(f^{-1})'(c)=\dfrac{1}{f'(a)} \Rightarrow (f^{-1})'(5)=\dfrac{1}{f'(e)}=\dfrac{e}{3}$.

2. Find $(f^{-1})'(c)$ for each of the following functions.

$a)\ f(x)=\ln(\sqrt{2x^3+3});c=0$ $b)\ f(x)=5e^{3x}+5;c=10$ $c)\ f(x)=x\ln x;\ c=2e^2$

$d)\ f(x)=\dfrac{x+6}{x-2};c=3$ $e)\ f(x)=\dfrac{3}{\sqrt{4x+1}};c=1$ $f)\ f(x)=\dfrac{e^x}{e^x+2};c=\dfrac{1}{3}$

Solution: In each case, first find a that satisfies the condition $f(a)=c$.

$a)\ f(a)=c \Rightarrow \ln(\sqrt{2a^3+3})=0 \Rightarrow \sqrt{2a^3+3}=e^0 \Rightarrow \sqrt{2a^3+3}=1$

$\Rightarrow 2a^3+3=1 \Rightarrow a^3=-1 \Rightarrow a=-1$

Besides, $f'(x)=\dfrac{1}{\sqrt{2x^3+3}}.(\dfrac{6x^2}{2\sqrt{2x^3+3}})=\dfrac{3x^2}{2x^3+3} \Rightarrow f'(-1)=3$.

Hence, $(f^{-1})'(c)=\dfrac{1}{f'(a)} \Rightarrow (f^{-1})'(0)=\dfrac{1}{f'(-1)}=\dfrac{1}{3}$.

b) Here, $f(a)=c \Rightarrow 5e^{3a}+5=10 \Rightarrow 5e^{3a}=5 \Rightarrow e^{3a}=1 \Rightarrow a=0$.

Besides, $f'(x)=15e^{3x} \Rightarrow f'(0)=15$. So, $(f^{-1})'(10)=\frac{1}{f'(0)}=\frac{1}{15}$.

c) Here, $a\ln a=2e^2 \Rightarrow \ln a^a=2e^2 \Rightarrow a^a=e^{2e^2} \Rightarrow a^a=(e^2)^{e^2} \Rightarrow a=e^2$.

(Note that for any positive real numbers x and y, $x^x=y^y \Rightarrow x=y$).

Besides, $f'(x)=\ln x+1 \Rightarrow f'(e^2)=\ln e^2+1=2+1=3$.

Hence, $(f^{-1})'(c)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(2e^2)=\frac{1}{f'(e^2)}=\frac{1}{3}$.

d) To find $(f^{-1})'(c)$ using the above theorem, first find a such that $f(a)=c$.

But, $f(a)=c \Rightarrow \frac{a+6}{a-2}=3 \Rightarrow 3a-6=a+6 \Rightarrow 2a=12 \Rightarrow a=6$

Besides, $f'(x)=\frac{-8}{(x-2)^2} \Rightarrow f'(6)=\frac{-8}{16}=-\frac{1}{2}$.

Thus, $(f^{-1})'(c)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(3)=\frac{1}{f'(6)}=-2$.

e) To find $(f^{-1})'(c)$ using the above theorem, first find a such that $f(a)=c$.

But, $f(a)=c \Rightarrow \frac{3}{\sqrt{4a+1}}=1 \Rightarrow \sqrt{4a+1}=3 \Rightarrow 4a+1=9 \Rightarrow a=2$

Besides, $f'(x)=-\frac{6}{(4x+1)^{\frac{3}{2}}} \Rightarrow f'(2)=-\frac{2}{9}$.

Thus, $(f^{-1})'(c)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(1)=\frac{1}{f'(2)}=-\frac{9}{2}$.

f) $\frac{e^a}{e^a+2}=\frac{1}{3} \Rightarrow 3a^a=e^a+2 \Rightarrow 2a^a=2 \Rightarrow a^a=1 \Rightarrow a=0$.

Hence, $(f^{-1})'(c)=\frac{1}{f'(a)} \Rightarrow (f^{-1})'(\frac{1}{3})=\frac{1}{f'(0)}=\frac{9}{2}$.

3. Find $(f^{-1})'(c)$ where

a) $f(x)=3+7\ln(x-4); c=3$ b) $f(x)=xe^{-3x}; c=e^{-3}$

c) $f(x)=5x^3+x; c=-6$ d) $f(x)=2+\ln(\sqrt{8x-15}); c=2$

e) $f(x)=x^7+x^5; c=-2$ f*) $f(x)=\ln\left(\frac{\pi}{2}+\tan^{-1}x\right); c=\ln\frac{\pi}{3}$

g) $f(x)=4e^x+x^3; c=4$ h) $f(x)=2\sqrt{\ln(e^x+\ln x)}; c=0$

Solution: First, find a such that $f(a)=c$.

a) $f(a)=c \Rightarrow 3+7\ln(a-4)=3 \Rightarrow \ln(a-4)=0 \Rightarrow a-4=e^0 \Rightarrow a=5$.

Therefore, we have $(f^{-1})'(3)=\frac{1}{f'(5)}=\frac{1}{7}$.

b) Here, $f(a)=c \Rightarrow ae^{-3a}=e^{-3} \Rightarrow a=1$. So, $(f^{-1})'(e^{-3})=\frac{1}{f'(1)}=-\frac{1}{2e^{-3}}$.

c) Here, $f(a)=c \Rightarrow 5a^3+a=-6 \Rightarrow 5a^3+a+6=0 \Rightarrow a=-1$.

Therefore, $(f^{-1})'(-6)=\frac{1}{f'(-1)}=\frac{1}{16}$.

d) $f(a)=c \Rightarrow 2+\ln(\sqrt{8a-15})=2 \Rightarrow \ln(\sqrt{8a-15})=0$

$$\Rightarrow \sqrt{8a-15}=e^0 \Rightarrow 8a-15=1 \Rightarrow 8a=16 \Rightarrow a=2$$

Therefore, we have $(f^{-1})'(2)=\frac{1}{f'(2)}=\frac{1}{4}$.

e) Here, $f(a)=c \Rightarrow a^7+a^5=-2 \Rightarrow a^7+a^5+2=0 \Rightarrow a=-1$.

Therefore, $(f^{-1})'(-2)=\frac{1}{f'(-1)}=\frac{1}{12}$.

f) Here, $\ln(\frac{\pi}{2}+\tan^{-1}a)=\ln\frac{\pi}{3} \Rightarrow \frac{\pi}{2}+\tan^{-1}a=\frac{\pi}{3} \Rightarrow \tan^{-1}a=-\frac{\pi}{6} \Rightarrow a=-\frac{1}{\sqrt{3}}$.

Besides, $f'(x)=\frac{1}{(\frac{\pi}{2}+\tan^{-1}x)(1+x^2)} \Rightarrow f'(-\frac{1}{\sqrt{3}})=\frac{9}{4\pi}$. So, $(f^{-1})'(\ln\frac{\pi}{3})=\frac{1}{f'(-\frac{1}{\sqrt{3}})}=\frac{4\pi}{9}$.

4. Let $F(x)=f(2g(x)), g(x)=f^{-1}(x), f(x)=x^4+x^3+1; 0\le x\le 3$. Find $F'(3)$.

Solution: By Chain rule, $F'(x)=2f'(2g(x))g'(x) \Rightarrow F'(3)=2f'(2g(3))g'(3)$.

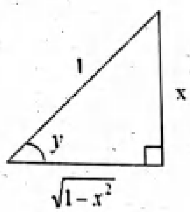
But, observe that $f(1)=3, g(3)=f^{-1}(3)=1$ and $g'(3)=(f^{-1})'(3)=\dfrac{1}{f'(1)}=\dfrac{1}{7}$.

So, $F'(3)=2f'(2g(3))g'(3)=2f'(2)\left(\dfrac{1}{7}\right)=\dfrac{88}{7}$.

5.6.4 Derivatives of Inverse Trigonometric Functions

Derivative of inverse sine function:

$y=\sin^{-1}x \Leftrightarrow \sin y = x$. Then, by differentiating both sides with respect to x, we get $\cos y\dfrac{dy}{dx}=1 \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{\cos y}$.



But from $\sin y = x$ using the right angle triangle, we have $\cos y=\sqrt{1-x^2}$,

$$\frac{dy}{dx}=\frac{1}{\cos y}=\frac{1}{\sqrt{1-x^2}} \Rightarrow \frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}.$$

Remark: Generally, $\dfrac{d}{dx}(\sin^{-1}f(x))=\dfrac{f'(x)}{\sqrt{1-(f(x))^2}}$

Examples: Find the derivative (dy/dx) of the following functions

a) $y=\sin^{-1}(3x^2)$ *b)* $5y^3=\sin^{-1}(e^{2x})$ *c)* $y=\sin^{-1}(\cos 3x^2)$

Solution:

a) $y=\sin^{-1}(3x^2) \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{\sqrt{1-(3x^2)^2}}(3x^2)'=\dfrac{6x}{\sqrt{1-9x^4}}$

b) $5y^3=\sin^{-1}(e^{2x}) \Rightarrow \dfrac{d}{dx}(5y^3)=\dfrac{d}{dx}[\sin^{-1}(e^{2x})]$

$$\Rightarrow 15y^2\frac{dy}{dx}=\frac{2e^{2x}}{\sqrt{1-e^{4x}}} \Rightarrow \frac{dy}{dx}=\frac{2e^{2x}}{15y^2\sqrt{1-e^{4x}}}$$

c) $y=\sin^{-1}(\cos 3x^2) \Rightarrow \dfrac{dy}{dx}=\dfrac{-6x\sin(3x^2)}{\sqrt{1-\cos^2(3x^2)}}=\dfrac{-6x\sin(3x^2)}{\sin(3x^2)}=-6x$

Derivative of inverse cosine (arc cosine) function:

$y=\cos^{-1}x \Leftrightarrow \cos y = x \Rightarrow -\sin y\dfrac{dy}{dx}=1 \Rightarrow \dfrac{dy}{dx}=\dfrac{-1}{\sin y}$. But from $\cos y = x$ using right triangle, we have $\sin y=\sqrt{1-x^2}$. So,

$$\frac{dy}{dx}=\frac{-1}{\sin y}=\frac{-1}{\sqrt{1-x^2}} \Rightarrow \frac{d}{dx}(\cos^{-1}x)=\frac{-1}{\sqrt{1-x^2}}.$$

Remark: Generally, $\dfrac{d}{dx}(\cos^{-1}f(x))=\dfrac{-f'(x)}{\sqrt{1-(f(x))^2}}$

Examples: Find the derivative $\dfrac{dy}{dx}$ of the following functions

a) $y=\cos^{-1}(2x)$ *b)* $y=\cos^{-1}(x^2)$ *c)* $y^2+1=\cos^{-1}(\sin x)$

Solution:

a) $y=\cos^{-1}(2x) \Rightarrow \dfrac{dy}{dx}=\dfrac{-2}{\sqrt{1-4x^2}}$

b) $y=\cos^{-1}(x^2) \Rightarrow \dfrac{dy}{dx}=\dfrac{-1}{\sqrt{1-x^4}}(x^2)'=\dfrac{-2x}{\sqrt{1-x^4}}$

c) $y^2+1=\cos^{-1}(\sin x) \Rightarrow \dfrac{d}{dx}(y^2+1)=\dfrac{d}{dx}(\cos^{-1}(\sin x))$

$$\Rightarrow 2y\frac{dy}{dx}=\frac{-\cos x}{\sqrt{1-\sin^2 x}}=\frac{-\cos x}{\cos x}=-1 \Rightarrow \frac{dy}{dx}=-\frac{1}{2y}$$

Derivative of inverse tangent function:

Here, $y=\tan^{-1}x \Leftrightarrow \tan y = x \Rightarrow \sec^2 y\dfrac{dy}{dx}=1 \Rightarrow \dfrac{dy}{dx}=\dfrac{1}{\sec^2 y}$. But from trigonometric identity we have $\sec^2 y = 1+\tan^2 y$ and $\tan y = x \Rightarrow \tan^2 y = x^2 \Rightarrow \sec^2 y = 1+x^2$. Thus,

$$\frac{d}{dx}(\tan^{-1}x)=\frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}.$$

Remark: Generally, $\dfrac{d}{dx}(\tan^{-1}f(x))=\dfrac{f'(x)}{1+(f(x))^2}$.

Examples: Find the derivative dy/dx of the following functions

a) $y=\tan^{-1}(3x^2)$ b) $3y^2=\tan^{-1}(\ln x)$ c) $y=\tan^{-1}(\ln x^2)$

Solution:

a) $y=\tan^{-1}(3x^2)\Rightarrow \frac{dy}{dx}=\frac{1}{1+(3x^2)^2}(3x^2)'=\frac{6x}{1+9x^4}$

b) $3y^2=\tan^{-1}(\ln x)\Rightarrow \frac{d}{dx}(3y^2)=\frac{d}{dx}[\tan^{-1}(\ln x)]$

$$\Rightarrow 6y\frac{dy}{dx}=\frac{1}{1+(\ln x)^2}(\ln x)'$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{6xy(1+(\ln x)^2)}$$

Derivative of inverse cotangent function:

$y=\cot^{-1}x \Leftrightarrow \cot y = x \Rightarrow \frac{d}{dx}(\cot y)=\frac{d}{dx}(x)$

$$\Rightarrow -\csc^2 y\frac{dy}{dx}=1\Rightarrow \frac{dy}{dx}=-\frac{1}{\csc^2 y}=-\sin^2 y$$

But from $\cot y = x$, we get $\sin y=\frac{1}{\sqrt{x^2+1}}\Rightarrow -\sin^2 y=-\frac{1}{x^2+1}$.

Thus, $\frac{d}{dx}(\cot^{-1}x)=\frac{dy}{dx}=-\frac{1}{1+x^2}$.

Generally, $\frac{d}{dx}(\cot^{-1}f(x))=\frac{-f'(x)}{1+(f(x))^2}$.

Examples: Find the derivative ($\frac{dy}{dx}$) of the following functions

a) $y=\cot^{-1}(x^2)$ b) $y=\cot^{-1}(\tan x^3)$ c) $y=\cot^{-1}(\sqrt{e^{-2x}-1})$

Solution:

a) $y=\cot^{-1}(x^2)\Rightarrow \frac{dy}{dx}=-\frac{1}{1+(x^2)^2}.(x^2)'=-\frac{2x}{1+x^4}$

b) $y=\cot^{-1}(\tan x^3)\Rightarrow \frac{dy}{dx}=-\frac{3x^2\sec^2(x^3)}{1+\tan^2(x^3)}=-\frac{3x^2\sec^2(x^3)}{\sec^2(x^3)}=-3x^2$

Derivative of Inverse secant (arc secant) Function:

$$y=\sec^{-1}x \Leftrightarrow \sec y = x \Rightarrow \frac{d}{dx}(\sec y)=\frac{d}{dx}(x)$$

$$\Rightarrow \sec y\tan y\frac{dy}{dx}=1 \Rightarrow \frac{dy}{dx}=\frac{1}{\sec y\tan y}$$

But from $\sec y = x$ using the right triangle, we have $\tan y=\sqrt{x^2-1}$. So,

$$\frac{d}{dx}(\sec^{-1}x)=\frac{dy}{dx}=\frac{1}{\sec y\tan y}=\frac{1}{x\sqrt{x^2-1}}.$$

Remark: Generally, $\frac{d}{dx}(\sec^{-1}f(x))=\frac{f'(x)}{f(x)\sqrt{(f(x))^2-1}}$

Examples: Find the derivative ($\frac{dy}{dx}$) of the following functions

a) $y=\sec^{-1}(2x)$ *b)* $y=\sec^{-1}(e^{2x})$ *c)* $y=\sec^{-1}(2x^4)$

Solution:

a) $y=\sec^{-1}(2x)\Rightarrow \frac{dy}{dx}=\frac{1}{2x\sqrt{1-4x^2}}(2x)'=\frac{2}{2x\sqrt{1-4x^2}}=\frac{1}{x\sqrt{1-4x^2}}$

b) $y=\sec^{-1}(e^{2x})\Rightarrow \frac{dy}{dx}=\frac{1}{e^{2x}\sqrt{e^{4x}-1}}(e^{2x})'=\frac{2e^{2x}}{e^{2x}\sqrt{e^{4x}-1}}=\frac{2}{\sqrt{e^{4x}-1}}$

c) $y=\sec^{-1}(2x^4)\Rightarrow \frac{dy}{dx}=\frac{1}{2x^4\sqrt{4x^8-1}}(2x^4)'=\frac{8x^3}{2x^4\sqrt{4x^8-1}}=\frac{4}{x\sqrt{4x^8-1}}$

Derivative of inverse cosecant Function:

$$y=\csc^{-1}x \Rightarrow \csc y = x \Rightarrow \frac{d}{dx}(\csc y)=\frac{d}{dx}(x)$$

$$\Rightarrow -\csc y\cot y\frac{dy}{dx}=1 \Rightarrow \frac{dy}{dx}=\frac{-1}{\csc y\cot y}$$

But from trigonometric relation, $\csc y = x \Rightarrow \sin y=\frac{1}{x},\ \cos y=\frac{\sqrt{x^2-1}}{x}$.

Thus, $\frac{d}{dx}(\csc^{-1}x)=\frac{dy}{dx}=\frac{-1}{\csc y\cot y}=\frac{-1}{x\sqrt{x^2-1}}$.

Remark: Generally, $\frac{d}{dx}(\csc^{-1} f(x)) = \frac{-f'(x)}{f(x)\sqrt{(f(x))^2 - 1}}$

Examples: Find the derivative $\frac{dy}{dx}$ of the following functions

a) $y = \csc^{-1}(3x^2)$ *b)* $y = \csc^{-1}(\cos x)$

Solution:

a) $y = \csc^{-1}(3x^2) \Rightarrow \frac{dy}{dx} = \frac{-6x}{3x^2\sqrt{9x^4 - 1}} = \frac{-2}{x\sqrt{9x^4 - 1}}$

b) $y = \csc^{-1}(\sec x) \Rightarrow \frac{dy}{dx} = \frac{-\sec^2 x \sin x}{\sec x\sqrt{\sec^2 x - 1}} = \frac{-\sec x \sin x}{\tan x} = -1$

5.6.5 Derivatives of Inverse Hyperbolic Functions

First: Recall what are hyperbolic functions:

Hyperbolic functions are functions which are complex analogue of trigonometric functions. They are defined using exponential functions as follow:

a) Hyperbolicsine : $\sinh x = \frac{e^x - e^{-x}}{2}$

b) Hyperboliccosine : $\cosh x = \frac{e^x + e^{-x}}{2}$

c) Hyperbolictangent : $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

d) Hyperboliccotangent : $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}; x \neq 0$

e) Hyperbolicsecant : $\sec hx = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

f) Hyperboliccosecant : $\csc hx = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}; x \neq 0$

Second: Observe Derivative of Hyperbolic Functions

Since hyperbolic functions are expressed in terms of e^x and e^{-x}, the formula for their derivatives are easily derived. For instance, let's see how to drive the derivative of hyperbolic sine function.

The derivative of $\sinh x$**:** $\frac{d}{dx}(\sinh x) = \frac{d}{dx}(\frac{e^x - e^{-x}}{2}) = \frac{e^x + e^{-x}}{2} = \cosh x$. The derivatives of the other hyperbolic functions are derived similarly.

$$\frac{d}{dx}(\sinh x) = \cosh x, \qquad \frac{d}{dx}(\sec hx) = -\sec hx \tanh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x, \qquad \frac{d}{dx}(\csc hx) = -\csc hx \coth x$$

$$\frac{d}{dx}(\tanh x) = \sec h^2 x, \qquad \frac{d}{dx}(\coth x) = -\csc h^2 x$$

Examples: Find the derivatives of

a) $y = \cosh(5x)$ *b)* $y^2 - 3y = \sinh(x^2)$ *c)* $6y = \tanh(e^{x^3})$ *d)* $y = \coth(\ln x)$

Solution:

a) $\frac{dy}{dx} = \frac{d}{dx}[\cosh(5x)] = \sinh(5x)\frac{d}{dx}(5x) = 5\sinh(5x)$

b) $\frac{d}{dx}(y^2 - 3y) = \frac{d}{dx}[\sinh(x^2)] \Rightarrow (2y-3)\frac{dy}{dx} = \cosh(x^2)\frac{d}{dx}(x^2) = 2x\cosh(x^2)$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\cosh(x^2)}{2y-3}$$

c) $6\frac{dy}{dx} = \frac{d}{dx}[\tanh(e^{x^3})] = 3x^2 e^{x^3} \sec h^2(e^{x^3}) \Rightarrow \frac{dy}{dx} = \frac{3x^2 e^{x^3} \sec h^2(e^{x^3})}{6}$

d) $\frac{dy}{dx} = \frac{d}{dx}[\coth(\ln x)] = -\csc h^2(\ln x)\frac{d}{dx}(\ln x) = \frac{-1}{x}\csc h^2(\ln x)$

Third: Examine Inverses of Hyperbolic Functions

Unlike trigonometric functions, hyperbolic functions are not periodic and most of them are one to one in their domain. For instance, the hyperbolic sine, tangent, cosecant and cotangent are one to one hyperbolic functions. Thus, these four functions have inverses in their domain. But the two hyperbolic functions: the hyperbolic cosine and hyperbolic secant are not one to one on the whole domain. However, by restricting their domain to be the set of non-negative real numbers, they will be one to one and for this restricted domain, they will have inverse functions. Now, let's see how to drive the inverse of hyperbolic functions.

The inverse of hyperbolic sine function: From hyperbolic sine function

$y = \sinh x = \dfrac{e^x - e^{-x}}{2}$, to find the inverse just interchange x and y in the formula and then solve for y in the same way as we find the inverse of algebraic functions.

$$y = \sinh x \Rightarrow x = \sinh y \Rightarrow x = \frac{e^y - e^{-y}}{2}$$

$$\Rightarrow 2x = e^y - e^{-y} \Rightarrow e^{2y} - 2xe^y - 1 = 0$$

Then, by letting $t = e^y$, we get $t^2 - 2xt - 1 = 0$. Hence, by quadratic formula ,we have $t = \dfrac{2x \pm \sqrt{4x^2 + 4}}{2} \Rightarrow t = x \pm \sqrt{x^2 + 1}$.

Then, from $t = e^y$, we have $e^y = x \pm \sqrt{x^2 + 1}$. Here, either $e^y = x - \sqrt{x^2 + 1}$ or $e^y = x + \sqrt{x^2 + 1}$. Since $e^y > 0$, $e^y = x - \sqrt{x^2 + 1}$ is invalid solution because $x < \sqrt{x^2 + 1}$ for all x which gives $x - \sqrt{x^2 + 1} < 0$ but this in turn implies $e^y < 0$ which is impossible. Therefore, $e^y = x + \sqrt{x^2 + 1}$ is the only valid solution.

Thus, $e^y = x + \sqrt{x^2 + 1} \Rightarrow y = \ln(x + \sqrt{x^2 + 1})$. Therefore, using $x = \sinh y \Rightarrow \sinh^{-1} x = y$, we have $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ on $(-\infty, \infty)$.

The inverse of hyperbolic tangent function:

$$y=\tanh x \Rightarrow x=\tanh y=\frac{\sinh y}{\cosh y}=\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}$$

$$\Rightarrow x=\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}} \Rightarrow x(e^{y}+e^{-y})=e^{y}-e^{-y}$$

$$\Rightarrow (x+1)e^{-y}=(1-x)e^{y} \Rightarrow e^{2y}=\frac{1+x}{1-x}$$

$$\Rightarrow y=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \Rightarrow \tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Therefore, $\tanh^{-1}x=\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$ on $(-1,1)$. Using similar derivation, we have

a) $\cosh^{-1}x=\ln(x+\sqrt{x^2-1})$ on $[1,\infty)$

b) $\coth^{-1}x=\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$ on $(-\infty,0)\cup(0,\infty)$

c) $\operatorname{sech}^{-1}x=\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$ on $(0,1]$

d) $\operatorname{csch}^{-1}x=\ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right)$ on $(-\infty,0)\cup(0,\infty)$

Fourth: Analyze Derivatives of Inverses Hyperbolic Functions

The technique to find the derivative of inverse hyperbolic functions is an easy task. As an example, let's find the derivative of inverse hyperbolic sine function and the others are derived similarly.

$$\frac{d}{dx}(\sinh^{-1}x)=\frac{d}{dx}[\ln(x+\sqrt{x^2+1})]=\frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}(x+\sqrt{x^2+1})$$

$$=\frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right)=\frac{1}{x+\sqrt{x^2+1}}\left(\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}\right)=\frac{1}{\sqrt{x^2+1}}$$

Similarly, the derivatives of the others are determined.

Theorem: (Derivative of inverse hyperbolic functions)

a) $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$ b) $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$

c) $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$ d) $\frac{d}{dx}(\operatorname{sec}h^{-1}x) = \frac{-1}{x\sqrt{1-x^2}}$

e) $\frac{d}{dx}(\operatorname{csc}h^{-1}x) = \frac{-1}{|x|\sqrt{1+x^2}}$ f) $\frac{d}{dx}(\coth^{-1} x) = -\frac{1}{1-x^2}$

Examples: Find the derivatives of

a) $y = \sinh^{-1}(5x)$ b) $y^3+1 = \tanh^{-1}(x^3)$ c) $y = \operatorname{sec}h^{-1}(e^{x^2})$

Solution:

a) $\frac{dy}{dx} = \frac{d}{dx}[\sinh^{-1}(5x)] = \frac{1}{\sqrt{25x^2+1}}\frac{d}{dx}(5x) = \frac{5}{\sqrt{25x^2+1}}$

b) $3y^2\frac{dy}{dx} = \frac{d}{dx}[\tanh^{-1}(x^3)] = \frac{1}{1-x^6}\frac{d}{dx}(x^3) = \frac{3x^2}{1-x^6} \Rightarrow \frac{dy}{dx} = \frac{x^2}{y^2(1-x^6)}$

c) $\frac{dy}{dx} = \frac{d}{dx}[\operatorname{sec}h^{-1}(e^{x^2})] = \frac{-1}{e^{x^2}\sqrt{1-e^{2x^2}}}\frac{d}{dx}(e^{x^2}) = \frac{-2x}{\sqrt{1-e^{2x^2}}}$

Finally: Practice Simplify Expressions with Hyperbolic Function

This section is devoted to develop your skills of simplifying functions involving compositions of trigonometric and hyperbolic functions and their inverses.

Examples:

1. Simplify the following expressions

a) $\sec(\sin^{-1}\sqrt{x})$ b) $\cos(2\sin^{-1}x)$ c) $\tan(\cos^{-1}\sqrt{1-x})$

d) $\sin(\cos^{-1}x)$ e) $\cos(2\tan^{-1}x)$ f) $\cos(\sin^{-1}3x)$

Solution:

a) Let $y = \sin^{-1}\sqrt{x}$ .Then, using right angle triangle,

$\sin y = \sqrt{x} \Rightarrow \cos y = \sqrt{1-x}$. Hence, $\sec(\sin^{-1}\sqrt{x}) = \sec y = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x}}$.

b) Suppose $2\sin^{-1}x = y \Rightarrow \sin\frac{y}{2} = x \Rightarrow \sin^2\frac{y}{2} = x^2$.

But $\cos^2\frac{y}{2} + \sin^2\frac{y}{2} = 1 \Rightarrow \cos^2\frac{y}{2} = 1-\sin^2\frac{y}{2} = 1-x^2$

Thus, $\cos(2\sin^{-1}x) = \cos y = \cos(\frac{y}{2}+\frac{y}{2}) = \cos^2\frac{y}{2} - \sin^2\frac{y}{2} = 1-x^2-x^2 = 1-2x^2$.

c) Suppose $y = \cos^{-1}(\sqrt{1-x})$.Then, $\cos y = \sqrt{1-x} \Rightarrow \sin y = \sqrt{x}$.

Hence, $\tan(\cos^{-1}\sqrt{1-x}) = \tan y = \dfrac{\sin y}{\cos y} = \sqrt{\dfrac{x}{1-x}}$.

d) Suppose $y = \cos^{-1}x$ .Then, $\cos y = x \Rightarrow \sin y = \sqrt{1-x^2}$. Hence,

$\sin(\cos^{-1}x) = \sin y = \sqrt{1-x^2}$

e) Suppose $2\tan^{-1}x = y$.

Then, $\tan\dfrac{y}{2} = x \Rightarrow \sin\dfrac{y}{2} = x\cos\dfrac{y}{2} \Rightarrow \sin^2\dfrac{y}{2} = x^2\cos^2\dfrac{y}{2}$.

Thus, $\cos(2\tan^{-1}x) = \cos y = \cos^2\dfrac{y}{2} - \sin^2\dfrac{y}{2} = \dfrac{1}{1+x^2} - \dfrac{x^2}{1+x^2} = \dfrac{1-x^2}{1+x^2}$

f) Let $y = \sin^{-1}3x \Rightarrow \sin y = 3x$.But $\sin^2 y + \cos^2 y \Rightarrow \cos y = \sqrt{1-9x^2}$.

Therefore, $\cos(\sin^{-1}3x) = \cos y = \sqrt{1-9x^2}$.

2. Simplify

a) $\cosh(\ln x)$ *b*) $\tanh(2\ln x)$ *c*) $\cosh(\sinh^{-1}x^2)$ *d*) $\tanh\left(\sinh^{-1}\left(\dfrac{1-x^2}{2x}\right)\right)$

Solution:

a) Since, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, we get $\cosh(\ln x) = \dfrac{e^{\ln x} + e^{\ln 1/x}}{2} = \dfrac{x + 1/x}{2} = \dfrac{x^2+1}{2x}$.

b) Since, $\tanh x = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$, we have

$$\tanh(2\ln x) = \frac{e^{2\ln x} - e^{-2\ln x}}{e^{2\ln x} + e^{-2\ln x}} = \frac{e^{\ln x^2} - e^{\ln 1/x^2}}{e^{\ln x^2} + e^{\ln 1/x^2}} = \frac{x^2 - 1/x^2}{x^2 + 1/x^2} = \frac{x^4-1}{x^4+1}$$

c) Let $y=\sinh^{-1}x^2$ and then $\sinh y = x^2$. But from hyperbolic identity, we have

$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \cosh^2 y = \sinh^2 y + 1 \Rightarrow \cosh y = \pm\sqrt{\sinh^2 y + 1}$.

Since hyperbolic function is always positive, we take only

$$\cosh y = \sqrt{\sinh^2 y + 1} \Rightarrow \cosh y = \sqrt{(x^2)^2+1} = \sqrt{x^4+1}$$

$$\Rightarrow \cosh(\sinh^{-1}x^2) = \sqrt{x^4+1}$$

d) Let $y=\sinh^{-1}\left(\frac{1-x^2}{2x}\right) \Rightarrow \sinh y = \frac{1-x^2}{2x}$. Besides, from hyperbolic identity,

$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \cosh^2 y = \sinh^2 y + 1$. Using $\sinh y = \frac{1-x^2}{2x}$, we get

$$\cosh^2 y = \sinh^2 y + 1 = \left(\frac{1-x^2}{2x}\right)^2 + 1 = \frac{1+2x^2+x^4}{4x^2} = \frac{(1+x^2)^2}{4x^2} \Rightarrow \cosh y = \frac{1+x^2}{2x}$$

Hence, $\tanh\left(\sinh^{-1}\left(\frac{1-x^2}{2x}\right)\right) = \tanh y = \frac{\sinh y}{\cosh y} = \frac{(1-x^2)/2x}{(1+x^2)/2x} = \frac{1-x^2}{1+x^2}$.

3. Find the numerical value of

a) $\sinh(\ln\sqrt{3})$ *b*) $\cosh(\ln 1/2)$ *c*) $\tanh(2\ln 5)$ *d*) $\coth(\ln 4)$

Solution:

a) Since, $\sinh x = \frac{e^x - e^{-x}}{2}$, we have

$$\sinh(\ln\sqrt{3}) = \frac{e^{\ln\sqrt{3}} - e^{-\ln\sqrt{3}}}{2} = \frac{e^{\ln\sqrt{3}} - e^{\ln 1/\sqrt{3}}}{2} = \frac{\sqrt{3}-1/\sqrt{3}}{2} = \frac{1}{\sqrt{3}}$$

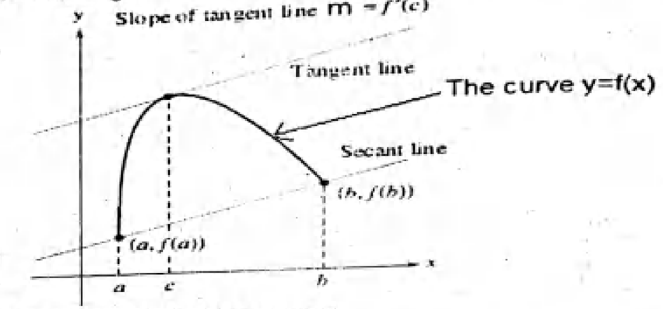
b) Since, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\cosh(\ln 1/2) = \frac{e^{\ln 1/2} + e^{-\ln 1/2}}{2} = \frac{e^{\ln 1/2} + e^{\ln 2}}{2} = \frac{5}{2}$

c) $$\tanh(2\ln 5) = \frac{e^{2\ln 5} - e^{-2\ln 5}}{e^{2\ln 5} + e^{-2\ln 5}} = \frac{e^{\ln 25} - e^{\ln 1/25}}{e^{\ln 25} + e^{\ln 1/25}} = \frac{25-1/25}{25+1/25} = \frac{624}{626} = \frac{312}{313}$$

d) $$\coth(\ln 4) = \frac{e^{\ln 4} + e^{-\ln 4}}{e^{\ln 4} - e^{-\ln 4}} = \frac{e^{\ln 4} + e^{\ln 1/4}}{e^{\ln 4} - e^{\ln 1/4}} = \frac{4+1/4}{4-1/4} = \frac{17/4}{15/4} = \frac{17}{15}$$

5.7 Geometric Interpretation of First Derivative

Tangent and normal lines: A line that intersects the graph of $y = f(x)$ exactly at two points is known as *secant line*. A line that intersects the graph of $y = f(x)$ exactly at one point is known as *tangent line* (Refer the diagram). The point where the tangent line touches the curve is called point of tangency.



The tangent line to a curve

As we see from the diagram, the secant line crosses the graph of $y = f(x)$ at $(a, f(a))$ and $(b, f(b))$. Thus, the equation of the secant line is given by

$$m = \frac{f(b)-f(a)}{b-a} = \frac{y-f(a)}{x-a} \Rightarrow y = \frac{f(b)-f(a)}{b-a}(x-a)+f(a).$$

But if we make the points $(a, f(a))$ and $(b, f(b))$ closer and closer, the slope of the secant line becomes closer and closer to the slope of the tangent line. And, thus, as a limiting value, we get the slope of the tangent line which gives us one geometric application of derivative. Geometrically, the first derivative of a function at a given point $x = a$ is interpreted as a slope of the tangent line at that point. That is $slope = m = f'(a)$. Once the slope is known, it is possible to determine the equation of the tangent line as follow.

$$m = f'(a) = \frac{y-f(a)}{x-a} \Rightarrow y - f(a) = f'(a)(x-a) \Rightarrow y = f'(a)(x-a)+f(a)$$

At this point, we can also see the concept of normal line to a curve. A line which is perpendicular to the tangent line at the point of *tangency* is known as *normal line* to the curve.

Since the product of the slope of two non-vertical perpendicular lines is -1, then slope of the normal line is $m=\frac{-1}{f'(a)}$ and its equation is

$$\frac{-1}{f'(a)}=\frac{y-f(a)}{x-a}\Rightarrow y-f(a)=\frac{-1}{f'(a)}(x-a)\Rightarrow y=\frac{-1}{f'(a)}(x-a)+f(a).$$

Examples:

1. Find the equations of the tangent and the normal lines to the graphs of

a) $f(x)=x^3-2x$ at $x=2$ b) $f(x)=e^{3x}\cos x$ at $x=0$

c) $f(x)=xe^{-x}+\sin 3x+5$ at $x=0$ d) $f(x)=x^3\ln(x^2+1)$ at $x=0$

e) $f(x)=\sec x\tan x$ at $x=\frac{\pi}{4}$ f) $f(x)=\frac{e^{4x}}{x^2+1}$ at $x=0$

Solution:

a) Here, $f'(x)=3x^2-2\Rightarrow m=f'(2)=10$. So, the equation of the tangent line is $l: y=f'(2)(x-2)+f(2)\Rightarrow y=10x-16$ and the equation of the normal line is

$$y=\frac{-1}{f'(2)}(x-2)+f(2)\Rightarrow 10y+x-42=0$$

b) Here, $f'(x)=3e^{3x}\cos x-e^{3x}\sin x\Rightarrow m=f'(0)=3$. So, the equation of the tangent line becomes $l: y=f'(0)(x-0)+f(0)\Rightarrow y=3x+1$ and the equation of the normal line is $y=-1/3x+1$.

c) $f'(x)=e^{-x}-xe^{-x}+3\cos 3x\Rightarrow m=f'(0)=1+3=4$.

So, the tangent line becomes $l: y-f(0)=4(x-0)\Rightarrow y-5=4x\Rightarrow y=4x+5$.

d) $f'(x)=3x^2\ln(x^2+1)+\frac{2x^4}{x^2+1}\Rightarrow m=f'(0)=0$. So, the equation of the tangent line becomes $l: y-f(0)=0\Rightarrow y=0$ and the normal line is $x=0$.

e) $y'=\sec x\tan^2 x+\sec^3 x\Rightarrow m=y'(\pi/4)=3\sqrt{2}$. So, the equation of the tangent line becomes $l: y-f(\pi/4)=3\sqrt{2}(x-\pi/4)\Rightarrow y=3\sqrt{2}(x-\pi/4)+\sqrt{2}$.

f) Here, $f'(x)=\frac{4e^{4x}(x^2+1)-2xe^{4x}}{(x^2+1)^2}\Rightarrow m=f'(0)=4$. So, the tangent line is $l: y=f'(0)(x-0)+f(0)\Rightarrow y=4x+1$ and the normal line is $y=-\frac{1}{4}x+1$.

2. Find the equation of the tangent line at the indicated point.

$a)\ f(x) = 8\tan(\frac{x}{4});$ at $x = \pi$ $\qquad b)\ f(x) = \sin x;$ at $x = \pi$

$c)\ f(x) = x\cos 3x;$ at $x = \pi$ $\qquad d)\ f(x) = \sin(1 + x^3);$ at $x = -1$

$e)\ f(x) = \sin(\sin x)$ at $x = \pi$ $\qquad f)\ f(x) = (2x+1)^{10}$ at $x = 0$

Solution

$a)$ Slope m $= f'(\pi) = 8\sec^2(\frac{\pi}{4}) = 4$

$$\Rightarrow l: y - f(\pi) = 4(x-\pi) \Rightarrow y = 4(x-\pi) + 8$$

$b)$ Slope m $= f'(\pi) = \cos(\pi) = -1$

$$\Rightarrow l: y - f(\pi) = -(x-\pi) \Rightarrow y = -x + \pi$$

$c)$ Slope m $= f'(\pi) = \cos(3\pi) - 3\pi\sin(3\pi) = -1$

$$\Rightarrow l: y - f(\pi) = -(x-\pi) \Rightarrow y = -x$$

$d)$ Slope m $= f'(-1) = 3(-1)^2\cos(0) = 3$

$$\Rightarrow l: y - f(-1) = 3(x+1) \Rightarrow y = 3(x+1)$$

$e)$ $f'(x) = \cos(\sin x)\cos x \Rightarrow m = f'(\pi) = -1 \Rightarrow l: y = -x + \pi$.

$f)$ $f'(x) = 20(2x+1)^9 \Rightarrow m = f'(0) = 20$

$$\Rightarrow l: y = m(x-0) + f(0) \Rightarrow y = 20x + 1$$

3. Find the equation of the tangent line to $y^2 - y = \tan^{-1}(4\ln x)$ at $(1,1)$.

Solution: Here, $y^2 - y = \tan^{-1}(4\ln x) \Rightarrow \dfrac{dy}{dx} = \dfrac{4}{x(1+16\ln^2 x)(2y-1)}$.

Slope: $m = \left.\dfrac{dy}{dx}\right|_{\substack{x=1\\y=1}} = \dfrac{4}{1.(1+16\ln^2 1)(2-1)} = 4$

Equation: $y - 1 = m(x-1) \Rightarrow y - 1 = 4(x-1) \Rightarrow y = 4x - 3$

4. Find the value of c if the line $4x - 9y = 0$ is tangent to the graph of the curve $f(x) = \dfrac{x^3}{3} + c$ in the first quadrant.

Solution: Here, the slope of the tangent line becomes $f'(x) = x^2$ at any point x.

But the line $4x - 9y = 0$ has slope $m = \frac{4}{9}$ which gives $x^2 = \frac{4}{9} \Rightarrow x = \pm\frac{2}{3}$.

Since we are interested to the tangent line in the first quadrant, we take only $x = \frac{2}{3}$. Hence, the point of tangency must satisfy both the equation of the tangent line and the curve. So, we have $y = \frac{4}{9}x = f(x) = \frac{x^3}{3} + c$ at the point of tangency.

That is $\frac{4}{9}\left(\frac{2}{3}\right) = \frac{(2/3)^3}{3} + c \Rightarrow c + \frac{8}{81} = \frac{8}{27} \Rightarrow c = \frac{16}{81}$.

5. Find all points on $f(x) = x^3 + x$ at which the tangent line has slope $m = 4$.

Solution: Here, $f'(x) = 3x^2 + 1 \Rightarrow m = f'(a) = 3a^2 + 1$. Then, if $m = 4$, we get $3a^2 + 1 = 4 \Rightarrow 3a^2 = 3 \Rightarrow a = \pm 1$. Thus, the points where the slope is $m = 4$ are $P = (1, f(1)) = (1,2)$, $Q = (-1, f(-1)) = (-1,-2)$.

6. Find the constant k if the line $y = 2x$ is tangent to the curve $f(x) = x^2 + k$.

Solution: Let the point of tangency be $(a, f(a))$. Here, $f'(x) = 2x$. Then, the slope of the tangent line at $(a, f(a))$ is $m = 2a$. But given that the tangent line at $(a, f(a))$ is $y = 2x$. So, $2a = 2 \Rightarrow a = 1$. Thus, since $(a, f(a))$ is also on the line $y = 2x$, $(a, f(a)) = (1,2) = (1, f(1)) \Rightarrow f(a) = f(1) = 2 \Rightarrow k + 1 = 2 \Rightarrow k = 1$.

7. If $y = 8x - 23$ is tangent line to the graph of f at $(5,17)$, find $f'(5)$.

Solution: At the point $(5,17)$, $f'(5)$ represents the slope of the tangent line, that is $f'(5) = 8$.

8. Suppose the curves $f(x) = x^2 + ax + b$ and $g(x) = cx - 3x^2$ have a common tangent line at $(2,0)$. Find the constants a, b and c.

Solution: Since f and g have a tangent at $(1,0)$, the point $(1,0)$ lies on both.

So, we have $\begin{cases} f(2) = 0 \Rightarrow 4 + 2a + b = 0 \Rightarrow 2a + b = -4 \\ g(2) = 0 \Rightarrow 2c - 12 = 0 \Rightarrow 2c = 12 \Rightarrow c = 6 \end{cases}$

Then, using $c = 6$, we get $g(x) = 6x - 3x^2$.

Again, to have a common tangent, the slopes must be equal.

That is $f'(2) = g'(2) \Rightarrow 4 + a = 6 - 12 \Rightarrow 4 + a = -6 \Rightarrow a = -10$.

Finally, $2a + b = -4 \Rightarrow 2(-10) + b = -4 \Rightarrow -20 + b = -4 \Rightarrow b = 16$.

Hence, the functions are $f(x) = x^2 - 10x + 16$ and $g(x) = 6x - 3x^2$.

Remarks (Vertical and Horizontal Tangents Lines):

i) If the first derivative of f at $x = a$ is zero, that is $f'(a) = 0$, then f has a horizontal tangent line with equation $l : y = f(a)$ and a vertical normal line given by $n : x = a$.

ii) If the first derivative of f at $x = a$ does not exist, that is

$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \pm\infty$, then f has a vertical tangent line with equation $l : x = a$ at $(a, f(a))$ and a horizontal normal line given by $n : y = f(a)$.

9. Find the equations of the tangent and normal lines to the graph of f.

a) $f(x) = 4x - x^2$ at x = 2

b) $f(x) = \sqrt[3]{x^2 - 1}$ at x = 1

b) $f(x) = x^{\frac{2}{3}}$ at x = 0

d) $f(x) = (2 - x)^{\frac{4}{5}}$ at x = 2

Solution:

a) Here, $f'(x) = 4 - 2x \Rightarrow m = f'(2) = 0$. So, f has a horizontal tangent line given by $l : y = f(2) = 4$ and a normal line given by $n : x = 2$.

b) Here, $f'(x) = \frac{2x}{3(x^2 - 1)^{\frac{2}{3}}} \Rightarrow f'(1) \to +\infty$. So, f has a vertical tangent line given by $l : x = 1$ and a horizontal normal line given by $n : y = 0$.

c) Here, $f'(x) = \frac{2}{3x^{1/3}} \Rightarrow f'(0) \to \pm\infty$. So, f has a vertical tangent given by $l : x = 0$ and its normal line is horizontal line which is $n : y = 0$.

10. Find the points where the functions have horizontal tangent lines

a) $f(x) = \frac{1}{4}x^4 - 8x + 16$

b) $f(x) = \frac{x^3}{x^2 - 3}$

c) $f(x) = 2x^3 - 9x^2 + 12x + 1$

d) $f(x) = \sqrt{8x - x^2}$

Solution: $f(x)$ will have horizontal tangent line at a point where $f'(x) = 0$.

a) $f'(x) = 0 \Rightarrow x^3 - 8 = 0 \Rightarrow x = 2$

b) $f'(x) = \dfrac{3x^2(x^2+1) - x^3(2x)}{(x^2+1)^2} = \dfrac{x^4 + 3x^2}{(x^2+1)^2} = 0 \Rightarrow x^2(x^2+3) = 0 \Rightarrow x = 0$

c) $f'(x) = 6x^2 - 18x + 12 = 0 \Rightarrow 6(x-2)(x-1) = 0 \Rightarrow x = 2, x = 1$

d) $f'(x) = \dfrac{4-x}{\sqrt{8x - x^2}} = 0 \Rightarrow 4 - x = 0 \Rightarrow x = 4$. Horizontal tangent at (4,4).

10. Find all the points where $f(x) = \sqrt{x^3 - 6x^2}$ has vertical tangent line.

Solution: The function will have vertical tangent line at the point where its first derivative is undefined. So, $f'(x) = \dfrac{3x^2 - 12x}{2\sqrt{x^3 - 6x^2}}$ and $f'(x)$ is undefined at the point where $x^3 - 6x^2 = 0 \Rightarrow x^2(x-6) = 0 \Rightarrow x = 0, 6$. Can we take 0?

5.8 Higher Order Derivatives

The derivative of f' if it exists is called the second derivative of f, denoted by f''. Then, it is defined as $f''(x) = \lim_{x \to a} \dfrac{f'(x) - f'(a)}{x - a}$ provided the limit exists.

In general, the n^{th} order derivative is $f^{(n)}(x) = \lim_{x \to a} \dfrac{f^{(n-1)}(x) - f^{(n-1)}(a)}{x - a}$. If this limit exist, we say that f is n times differentiable. The n^{th} derivatives $f^{(n)}$ for $n \geq 2$ are called *higher order derivatives* of f.

Hints: Hints to find n^{th} - term derivative formula:

1. Observe pattern of exponents or powers
2. Observe pattern of coefficients
3. Change of signs of coefficients

Examples: Find the formula for the n^{th} derivatives of the following functions

$a)\ f(x)=e^{2x}$ $b)\ f(x)=3^{-2x}$ $c)\ f(x)=\dfrac{1}{x}$ $d)\ f(x)=\dfrac{1}{2x+1}$

$e)\ f(x)=xe^{x}$ $f)\ f(x)=\ln x$ $g)\ f(x)=e^{-3x}$ $h)\ f(x)=\sin 3x$

$i)\ f(x)=xe^{-x}$ $j)\ f(x)=e^{3x}+x\ln x$ $k)\ f(x)=\dfrac{1}{x}+x\ln x$

Solution:

$a)\ f'(x)=2e^{2x},\ f''(x)=2^2e^{2x},\ f'''(x)=2^3e^{2x},\ f^{(4)}(x)=2^4e^{2x}$

$\therefore f^{(n)}(x)=2^ne^{2x},\ \forall n\in N$

$b)\ f'(x)=-2(\ln 3).3^{-2x},\ f''(x)=(-2)^2(\ln 3)^2.3^{-2x},\ f^{(3)}(x)=(-2)^3(\ln 3)^3.3^{-2x}.$

$\therefore f^{(n)}(x)=(-2)^n(\ln 3)^n.3^{-2x}=(-1)^n.2^n.(\ln 3)^n.3^{-2x}$

$c)\ f(x)=\dfrac{1}{x}=x^{-1},\ f'(x)=(-1)x^{-2},\ f^{(2)}(x)=2x^{-3},\ f^{(3)}(x)=(-1)6x^{-4},$

$f^{(4)}(x)=24x^{-5},\ f^{(5)}(x)=-120x^{-6}\ \therefore f^{(n)}(x)=(-1)^nn!x^{-(n+1)},\ \forall n\in N$

$d)\ f(x)=\dfrac{1}{2x+1}=(2x+1)^{-1},f'(x)=-2(2x+1)^{-(1+1)},f''(x)=2(2^2)(2x+1)^{-(2+1)}$

$f^{(3)}(x)=-6(2)^3(2x+1)^{-(3+1)},\ 6=3!.\ \therefore f^{(n)}(x)=(-1)^nn!(2^n)(2x+1)^{-(n+1)}$

$e)\ f'(x)=xe^x+e^x=(x+1)e^x,\ f''(x)=(x+1)e^x+e^x=(x+2)e^x$

$f'''(x)=(x+3)e^x,\ f^{(4)}(x)=(x+4)e^x.\ \therefore f^{(n)}(x)=(x+n)e^x,\ \forall n\in N$

$f)\ f'(x)=\dfrac{1}{x}=x^{-1},\ f''(x)=(-1)x^{-2};\ f^{(3)}(x)=2x^{-3},f^{(4)}(x)=(-1)6x^{-4}$

$f^{(5)}(x)=24x^{-5},f^{(6)}(x)=-120x^{-6}\ \therefore f^{(n)}(x)=(-1)^{n+1}(n-1)!x^{-n}$

5.9 Implicit Differentiation

So far, we have seen how to find the derivatives of functions where y is defined explicitly in the form $y = f(x)$. But what will be the derivative of functions where y is not expressed in terms of x or where it is impossible to express y as a function of x. The derivatives of such functions is obtained by a method called *Implicit differentiation* and denoted by $y' = dy/dx$. To find dy/dx for implicitly given functions, simply take the derivative both sides with respect to x and multiply the derivative of each term involving y by dy/dx (since y is supposed to be a function of x). Besides, apply product, quotient and chain rules of derivative in the usual way. Finally, solve for dy/dx.

Examples: Find the derivatives or dy/dx using implicit differentiation.

a) $x^2 + y^2 = 25$ *b)* $y^5 + 2y + x = 0$ *c)* $x^2y^2 = 2x + y$

d) $y^2 = \sin(xy)$ *e)* $e^{xy} = -xy$ *f)* $\ln(x+y) = ye^x$

Solution:

a) $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

b) $\frac{d}{dx}(y^5 + 2y + x) = 0 \Rightarrow 5y^4\frac{dy}{dx} + 2\frac{dy}{dx} + 1 = 0 \Rightarrow \frac{dy}{dx} = \frac{-1}{5y^4 + 2}$

c) $2xy^2 + 2x^2y\frac{dy}{dx} = 2 + \frac{dy}{dx} \Rightarrow (2x^2y - 1)\frac{dy}{dx} = 2 - 2xy^2 \Rightarrow \frac{dy}{dx} = \frac{2 - 2xy^2}{2x^2y - 1}$

d) $2y\frac{dy}{dx} = \cos(xy)(y + x\frac{dy}{dx}) \Rightarrow \frac{dy}{dx} = \frac{y\cos(xy)}{2y - x\cos(xy)}$

e) $e^{xy}\frac{d}{dx}(xy) = -y - x\frac{dy}{dx} \Rightarrow e^{xy}(y + x\frac{dy}{dx}) = -y - x\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

f) $\frac{1}{x+y}\frac{d}{dx}(x+y) = e^x\frac{dy}{dx} + ye^x \Rightarrow \frac{dy}{dx} = \frac{ye^x(x+y) - 1}{1 - e^x(x+y)}$

Equation of Tangent Lines to the graph of Implicit Functions:

As we have seen earlier $f'(a)$ is the slope of the tangent line to the graph of f at $(a, f(a))$. In a similar fashion, if we use implicit differentiation, the slope of the tangent line at (x_0, y_0) becomes $m = \left.\frac{dy}{dx}\right|_{\substack{x=x_0 \\ y=y_0}}$. This means to obtain the equation of the tangent line to the graph of implicit function at (x_0, y_0):

First: Find the formula of dy/dx using implicit differentiation.

Second: Evaluate dy/dx using the point (x_0, y_0).

Third: Write the equation using $y - y_0 = m(x - x_0)$ where $m = \left.\frac{dy}{dx}\right|_{\substack{x=x_0 \\ y=y_0}}$

Examples:

1. Find the slope of the tangent line to the curves at the given point

$a)$ $2e^{x^2y} = x$; at (2,0) $\quad b)$ $x^2 + y^2 = y$; at (0,1)

$c)$ $7y^4 + x^3y + x = 4$; at (4,0) $\quad d)$ $x^3 + 2xy = 5$; (1,2)

Solution:

$a)$ $2e^{x^2y} = x \Rightarrow \frac{dy}{dx} = \frac{1-4xye^{x^2y}}{2x^2e^{x^2y}} \Rightarrow m = \left.\frac{dy}{dx}\right|_{\substack{x=2 \\ y=0}} = \frac{1}{8}$

$b)$ $x^2 + y^2 = y \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y-1} \Rightarrow m = \left.\frac{dy}{dx}\right|_{\substack{x=0 \\ y=1}} = 0$

$c)$ $7y^4 + x^3y + x = 4 \Rightarrow \frac{dy}{dx} = \frac{-3x^2y-1}{28y^3+x^3} \Rightarrow m = \left.\frac{dy}{dx}\right|_{\substack{x=4 \\ y=0}} = \frac{-1}{64}$

2. Find the equations of the tangent and normal lines at the given point.

$a)$ $y^2 = 5x^4 - x^2$ at (1,2) $\quad b)$ $\sin(x+y) = 2x$ at $(0, \pi)$ $\quad c)$ $x^2 + y^2 = 25$ at (-3,4)

$d)$ $xe^y + 2 = y + x^2$ at (2,0) $\quad e)$ $x^{2/3} + y^{2/3} = 5$ at (8,1) $\quad f^*)$ $\sin(xy) = y$; $\left(\frac{\pi}{2}, 1\right)$

$g)$ $xy^2 = 18$; $(2,-3)$ $\quad h)$ $16x^4 + y^4 = 32$; (1,2) $\quad i)$ $xe^y + ye^x = 1$; (0,1)

$j)$ $y^2e^x + 3xy = 1$; (0,1) $\quad k^*)$ $x^2 + y^2 = 1$; (1,0)

Solution:

a) $\frac{d}{dx}(y^2)=\frac{d}{dx}(5x^4-x^2)\Rightarrow 2y\frac{dy}{dx}=20x^3-2x\Rightarrow\frac{dy}{dx}=\frac{10x^3-x}{y}$.

So, the slope of the tangent line is given by $m=\left.\frac{dy}{dx}\right|_{\substack{x_0=1\\y_0=2}}=\frac{9}{2}$.

Hence, the equation is $y=m(x-x_0)+y_0\Rightarrow y=\frac{9}{2}(x-1)+2\Rightarrow 2y=9x-5$ and

the normal line is $y=\frac{-1}{m}(x-x_0)+y_0\Rightarrow y=\frac{-2}{9}(x-1)+2\Rightarrow 2x+9y-20=0$.

b) $\frac{dy}{dx}=\frac{2-\cos(x+y)}{\cos(x+y)}\Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=\pi}}=-3$. Hence, the equation of the tangent is

$l: y-\pi=-3(x-0)\Rightarrow l: y=-3x+\pi$ and the normal line is $x-3y+3\pi=0$.

c) $\frac{dy}{dx}=\frac{-y}{x}\Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=-3\\y=4}}=\frac{3}{4}$. Hence, the tangent line is

$l: y-4=\frac{3}{4}(x+3)\Rightarrow l: 3x-4y+25=0$ and the normal line is $4x+3y=0$.

d) $\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}\Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=8\\y=1}}=-\frac{1}{2}$. Hence, the tangent line is

$l: y-y_0=m(x-x_0)\Rightarrow y-1=-\frac{1}{2}(x-8)\Rightarrow l: x+2y-10=0$ and the normal

line is $y=2x-25$.

d) $xe^y+2=y+x^2\Rightarrow\frac{dy}{dx}=\frac{2x-e^y}{xe^y-1}\Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=0}}=\frac{4-1}{2-1}=3$

Hence, the equation of the tangent line is $l: y-0=3(x-2)\Rightarrow l: y=3x-6$ and

that of the normal line is $l: y=-\frac{1}{3}(x-2)$.

e) $x^{2/3}+y^{2/3}=5\Rightarrow\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}\Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=8\\y=1}}=-\sqrt[3]{\frac{1}{8}}=-\frac{1}{2}$.

Hence, the equation of the tangent line is $l: y-1=-\frac{1}{2}(x-8)=-\frac{1}{2}x+5$ and that of the normal line is $l: y-1=2(x-8)=2x-15$.

f) $\frac{dy}{dx}=\frac{-y\cos xy}{x\cos xy-1} \Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=\pi/2\\y=1}}=0$. Since the slope is zero the curve has horizontal tangent and vertical normal lines at $(\pi/2,1)$. Hence, the tangent line is $l: y=1$ and the normal line is $n: x=\pi/2$.

g) $xy^2=18 \Rightarrow \frac{dy}{dx}=\frac{-y}{2x} \Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=2\\y=-3}}=\frac{3}{4}$. Hence, the equation of the tangent line is $l: y-b=m(x-a) \Rightarrow y+3=\frac{3}{4}(x-2) \Rightarrow l: 3x-4y-18=0$.

h) $16x^4+y^4=32 \Rightarrow \frac{dy}{dx}=\frac{-16x^3}{y^3} \Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=1\\y=2}}=-2$.

Hence, the equation of the tangent line is
$l: y-b=m(x-a) \Rightarrow y-2=-2(x-1) \Rightarrow l: 2x+y-4=0$.

i) $xe^y+ye^x=1 \Rightarrow \frac{dy}{dx}=\frac{-(e^y+ye^x)}{xe^y+e^x} \Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}}=-e-1$.

Hence, the equation of the tangent line is
$l: y-b=m(x-a) \Rightarrow y-1=(-e-1)(x-0) \Rightarrow l: y=-(e+1)x+1$.

j) $y^2e^x+3xy=1;(0,1)$ $\frac{dy}{dx}=\frac{-(y^2e^x+3y)}{2ye^x+3x} \Rightarrow m=\left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}}=-2$

Hence, the equation of the tangent line is $l: y-1=-2(x-0) \Rightarrow y=-2x+1$.

k) $x^2+y^2=1 \Rightarrow \frac{dy}{dx}=\frac{-x}{y}$. But here, $\frac{dy}{dx}=\frac{-x}{y}$ is undefined at the point $(1,0)$.

This means the tangent line at this point is a vertical line which is $x=1$.

3. Find points on $x^3+y^3=3xy$ where the tangent line is horizontal.

5.10 Applications of Derivatives

5.10.1 Solving Related Rates (Rate of change) Problems

Time Dependent Variables:

The change of a variable with respect to time is called time rate of change or simply rate of change. One important application of derivative is its power to calculate the magnitude of change of a variable with respect to time.

Notation: In differential calculus, the rate of change of a variable V with respect to time t is denoted by $\frac{dV}{dt}$.

Formula for Related Rates:

If two time dependent variables x and y are related to each other by some function f, then their rates of change are also related to each other. But the formula for the related rates depends on the type of the relation. That means based on whether the relation is explicit or implicit.

Explicit Relation:

In the relation of x and y, if y is expressed or we can solve for y in terms of x in the form: $y = \mathbf{R}(x)$, the relation is explicit relation. In such case, we have the following related rate formula.

Related Rate Formula: $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$ (Used for explicitly related variables)

Implicit Relation: In the relation of x and y, if we cannot solve for y in terms of x, then the relation is said to be implicit relation. In this case, the relation is expressed in the form: $\mathbf{R}(x,y) = C$ (some constant).

For this type of relation, we have the following related rate formula.

Related Rate Formula: $\frac{d\mathbf{R}}{dx}.\frac{dx}{dt} + \frac{d\mathbf{R}}{dy}.\frac{dy}{dt} = 0$ (Used for implicit relation)

Remark: Sometimes, when a quantity is a function of two or more quantities, we may face problems involving more than two rates. In such case, assume all variables as functions of time and relate each time derivatives.

Procedures: To solve problems concerning related rates;

Step-1: Identify the relation (equation). That means the first task is to identify how the variables with the known (given) rate of change and with unknown (required) rate of change are connected. Here, the relation is obtained by using basic relations and formula from geometry.

Step-2: Compute preliminary data or value of a variable.

Usually, in related rate problems, the value of one variable and one rate of change are given such that the rate of another variable is asked. So, in this step, the task is to find the value of the variable from the mathematical relation obtained in step-1.

Step-3: Use Related Rate Formula.

Based on the type of relation $\frac{dy}{dt}=\frac{dy}{dx}.\frac{dx}{dt}$ or $\frac{d\mathbf{R}}{dx}.\frac{dx}{dt}+\frac{d\mathbf{R}}{dy}.\frac{dy}{dt}=0$.

To use these formula, start by finding $\frac{dy}{dx}$ or $\frac{d\mathbf{R}}{dx}$ & $\frac{d\mathbf{R}}{dy}$ from the relation.

Once you identified the given and required rates, use one of the above related rate formula and solve for the unknown rate.

Remark: Frequently used Geometric Formula in rate problems

Many of the rate problems that you will face in this level uses the known geometric formula related to: Area, Perimeter, Volume, Surface Area, Pythagoras Theorem, Distance, Cosine laws and others.

Examples:

1. Suppose x and y are related by $y^2=x^3$ where $x>0, y>0$. Then, using this relation, find;

a) The rate of change of y when $x=4$ and its rate of change is $2\,\text{cm}^2/\text{sec}$.

b) The rate of change of x when $y=27$ and its rate of change is $18\,\text{cm}^3/\text{sec}$.

Solution: To make the idea clear, let us follow the above steps.

a) We need $\frac{dy}{dt}$ when $x=4$ and $\frac{dx}{dt}=2$.

Step-1: Identify the relation (equation).

Here, the variable are implicitly related by: $y^2=x^3$.

Step-2: Compute value of a variable.

We need to solve the problem when $x = 4$. But we do not have the value of y when $x = 4$. So, use $x = 4$ in the relation $y^2 = x^3$ to know the value of y.

That is $y^2 = x^3 \Rightarrow y^2 = 64 \Rightarrow y = \pm 8$. Since $y > 0$, only $y = 8$ is valid.

Step-3: Use Related Rate Formula.

Since we have implicit relation, we use the formula: $\frac{d\mathbf{R}}{dx}.\frac{dx}{dt} + \frac{d\mathbf{R}}{dy}.\frac{dy}{dt} = 0$.

Express the relation in the form $\mathbf{R}(x, y) = C$.

That is $y^2 = x^3 \Rightarrow y^2 - x^3 = 0 \Rightarrow \mathbf{R}(x, y) = y^2 - x^3 = 0$.

Then we have $\frac{d\mathbf{R}}{dx} = -3x^2$, $\frac{d\mathbf{R}}{dy} = 2y$.

Besides, $\frac{d\mathbf{R}}{dx}.\frac{dx}{dt} + \frac{d\mathbf{R}}{dy}.\frac{dy}{dt} = 0 \Rightarrow -3x^2\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = \frac{3x^2}{2y}\frac{dx}{dt}$

Therefore, $\frac{dy}{dt} = \frac{3(16)}{2(8)}(2) = 6\,\text{cm}^3/\text{sec}.$

b) We need $\frac{dx}{dt}$ when $y = 27$ and $\frac{dy}{dt} = 18$.

Step-1: Identify the relation (equation).

Here, the variable are implicitly related by: $y^2 = x^3$.

Step-2: Compute value of a variable.

Here, $y^2 = x^3 \Rightarrow x^3 = 27^2 \Rightarrow x = \sqrt[3]{27 \times 27} = 9$.

Step-3: Use Related Rate Formula.

That is $y^2 = x^3 \Rightarrow y^2 - x^3 = 0 \Rightarrow \mathbf{R}(x, y) = y^2 - x^3 = 0$.

Then we have $\frac{d\mathbf{R}}{dx} = -3x^2$, $\frac{d\mathbf{R}}{dy} = 2y$.

Besides, $\frac{d\mathbf{R}}{dx}.\frac{dx}{dt} + \frac{d\mathbf{R}}{dy}.\frac{dy}{dt} = 0 \Rightarrow -3x^2\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \Rightarrow \frac{dx}{dt} = \frac{2y}{3x^2}\frac{dy}{dt}$

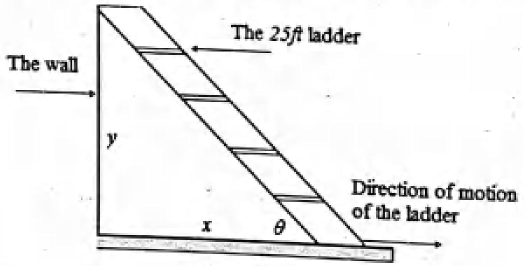
Therefore, $\frac{dx}{dt} = \frac{2(27)}{3(81)}(18) = 4\,\text{cm}^2/\text{sec}.$

Hey! Note that this example is simple in that the relation is given. This is just to make the idea clear but in most cases, the relation or the equation connecting the variables is developed by the students. *However, if you understand both parts of this example, you have got the hidden secret of solving related rate problems. If not you have missed for the second time what you have already missed. So, what are you waiting for?*

2. A ladder of $25\,ft$ long leans against a wall. If the boy is pulling the lower end of the ladder away from the wall at a rate of $4\,ft/\min$, at the instant when the lower end is $7\,ft$ away from the wall,

a) Find the rate at which the top end of the ladder falling to the ground.
b) At what rate angle θ between the ladder and the ground is changing?
c) How fast is the area of the triangle changing?
d) Give the rate of change of the perimeter.

Solution: Here, we assume that the wall and the ground are at right angle and the ladder is the hypotenuse of the right angled triangle as shown below.



Then, from Pythagoras theorem, x and y are related by the equation $x^2 + y^2 = 625$. At the instant when $x = 7\,ft$, the top of the wall is at $y = 24\,ft$.

Besides, $x^2 + y^2 = 625 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$.

Thus, $\frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt} = -\frac{x}{y}.\frac{dx}{dt} = -\frac{7}{24} \times 4 = -\frac{7}{6}$. This means that the ladder is falling to the ground at a rate of $\frac{7}{6}\,ft/\min$.

The negative sign shows that the height of the wall is decreasing by that rate. In general, the time rate of change is negative if a quantity is decreasing through time and is positive if it is increasing.

b) $\tan\theta = \frac{y}{x} \Rightarrow \frac{d}{dt}(\tan\theta) = \frac{d}{dt}(\frac{y}{x}) \Rightarrow \sec^2\theta\frac{d\theta}{dt} = \frac{1}{x^2}(x\frac{dy}{dt} - y\frac{dx}{dt})$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\cos^2\theta}{x^2}[x\frac{dy}{dt} - y\frac{dx}{dt}] = \frac{49}{625(49)}(-\frac{49}{6} - 96) = -\frac{1}{6}ft/\min$$

3. Suppose a spherical balloon grows in such a way that the volume V after t seconds is $V = 4\pi\sqrt{t}cm^3$. How fast is the radius changing after 81 seconds?

Solution: We know that the volume of a sphere is given by $V = \frac{4}{3}\pi r^3$.

Now, take time derivatives of this formula and the given $V = 4\pi\sqrt{t}cm^3$.

That is as $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$, $\frac{dV}{dt} = \frac{2\pi}{\sqrt{t}}$.

Equating the two values of $\frac{dV}{dt}$, we have $4\pi r^2\frac{dr}{dt} = \frac{2\pi}{\sqrt{t}} \Rightarrow \frac{dr}{dt} = \frac{1}{2r^2\sqrt{t}}$.

Besides, when $t = 81$ sec,using $V = 4\pi\sqrt{t}cm^3$, we have $V = 36\pi$.

Then, $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{4}{3}\pi r^3 = 36\pi \Rightarrow r^3 = 27 \Rightarrow r = 3cm$.

Therefore, $\frac{dr}{dt} = \frac{1}{2r^2\sqrt{t}} = \frac{1}{2(3)^2\sqrt{81}} = \frac{1}{162}$ cm/sec.

4. Suppose the radius of a spherical balloon is shrinking at a rate of $\frac{1}{2}$ cm/sec. How fast is the volume decreasing when the radius is 4cm?

Solution: $V = \frac{4}{3}\pi r^3 \Rightarrow dV/dt = 4\pi r^2 dr/dt = 4\pi(16)(-1/2cm/s) = -32\pi cm^3/s$

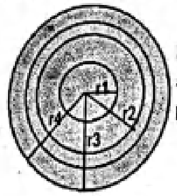
5. Gas is being pumped into a spherical balloon at a rate of $3cm^3/\sec$. Find the rate at which the radius is changing at the instant when the diameter is 12cm.

Solution: Here,

$$V = \frac{4}{3}\pi r^3 \Rightarrow dV/dt = 4\pi r^2 dr/dt \Rightarrow dr/dt = \frac{dV/dt}{4\pi r^2} = \frac{3}{4\pi(36)} = \frac{1}{48\pi}cm/s$$

6. Water leaking onto a floor creates a circular pool with an area increasing at a rate of $3cm^2/\sec$. How fast is the radius of the water pool increasing when the radius is 10cm?

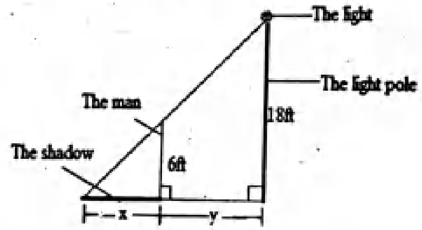
Solution: The condition of the problem looks like as shown below.



Here, $A=\pi r^2 \Rightarrow dA/dt = 2\pi r dr/dt \Rightarrow dr/dt = \dfrac{dA/dt}{2\pi r} = \dfrac{3}{20\pi} cm/\sec$.

7. A man $6\,ft$ tall is moving along a straight line at a rate of $3\,ft/\sec$ toward a streetlight pole of height $18\,ft$. At what rate is the length of his shadow changing? How fast is the tip of his shadow moving?

Solution: Let x be length of shadow and y distance of the man as shown.



Here, we are required to find $\dfrac{dx}{dt}$ at the instant when $\dfrac{dy}{dt} = -3\,ft/\sec$.(Here we used negative sign because from the physical condition of the problem, the distance between the man and the pole is decreasing). But remember that if the man was moving away from the pole, the positive rate would be used. Now using similar triangles, we have $\dfrac{6}{x} = \dfrac{18}{x+y} \Rightarrow x = \dfrac{y}{2}$.

But $x = \dfrac{y}{2} \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{2}\dfrac{dy}{dt} \Rightarrow \dfrac{dx}{dt} = \dfrac{1}{2}(-3) = -\dfrac{3}{2}\,ft/\sec$. For the second part of the problem, let's use this result as follow.

As we see from the diagram, the tip of the shadow is at a distance of $S = x + y$ from the pole. Thus, its rate of motion is given by $\frac{dS}{dt} = \frac{dx}{dt} + \frac{dy}{dt}$.

Therefore, using the first result, $\frac{dS}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -\frac{3}{2} - 3 = -\frac{9}{2}\, ft/\sec$.

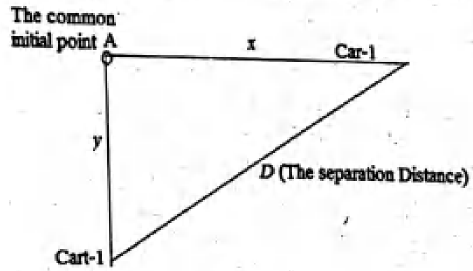
8. If the radius of a circle is increasing at a rate of $3/2cm/\sec$, find the rate at which the area is increasing at the instant when the diameter is $12cm$.

Solution: Here, $A = \pi r^2 \Rightarrow dA/dt = 2\pi r dr/dt$. Besides, when the diameter is $12cm$, the radius becomes $6cm$.

Hence, $dA/dt = 2\pi r dr/dt = 2\pi(6)(3/2) = 18\pi cm^2/\sec$.

9. Two cars start moving from the same point. One travels south at $60km/h$ and the other travels west at $25km/h$. Find the rate how the distance between the cars is changing after $2hrs$. Is the distance increasing or decreasing?

Solution: Let the distance moved by each car be x and y as shown below.



Then, from Pythagoras Theorem, we have $D^2 = x^2 + y^2$.

From these distances, the separation between the cars is calculated using Pythagoras Theorem: $D^2 = x^2 + y^2 \Rightarrow D = \sqrt{(120)^2 + (50)^2} = 130\text{km}$.

Then, the rate of change of distance becomes,

$$D^2 = x^2 + y^2 \Rightarrow 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \Rightarrow \frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

Besides, using the equation of motion and the given speeds of the cars, the distances moved after $2hrs$ are computed as follow:

That is $x = v_1 t = 60(2) = 120\text{km}$, $y = v_2 t = 25(2) = 50\text{km}$.

Besides, in motion problems, speed means rate of change of distance in a given time. So, the rates are $\frac{dx}{dt} = 60\text{km/h}, \frac{dy}{dt} = 25\text{km/h}$.

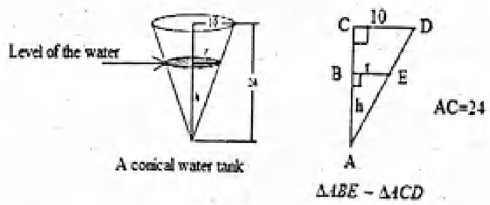
Again, use related rate formula for implicit functions:

$$\frac{dD}{dt} = \frac{1}{D}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \Rightarrow \frac{dD}{dt} = \frac{1}{130}\left(120(60) + 50(25)\right) = 65\text{km/h} .$$

Since the rate of change is positive, the distance between them is increasing.

10. A conical water tank with vertex down has radius $10ft$ at the top with a height of $24ft$. If water flows into the tank at $20ft^3/\min$, how fast is the depth of the water increasing when the level of the water is $16ft$ deep?

Solution: Consider the diagram.



The volume of the water at height h and radius of r is given by $V = \frac{1}{3}\pi r^2 h$.

Now using similarity of triangles, as shown in the diagram, we have $\Delta ABE \sim \Delta ACD \Rightarrow \frac{AB}{AC} = \frac{BE}{CD} \Rightarrow \frac{h}{24} = \frac{r}{10} \Rightarrow r = \frac{5}{12}h$.

Thus, $V = \frac{1}{3}\pi r^2 h = \frac{25\pi}{432}h^3 \Rightarrow \frac{dV}{dt} = \frac{25\pi}{144}h^2\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{144}{25\pi h^2}\frac{dV}{dt}$.

So, when $\frac{dV}{dt} = 20ft^3/\min,\ h = 16ft,\ \frac{dh}{dt} = \frac{144}{25\pi(16)^2}.20 = \frac{9}{20\pi}ft/\min$

11. Suppose two sides of a triangle are 15ft and 20 ft long. How fast is the third side changing when the angle between the given sides is $\theta = \frac{\pi}{3}$ and increasing at a rate of $\frac{\pi}{90}$ radians per second?

Solution: Apply cosine law to get $\frac{dz}{dt} = 600.\frac{\sin\frac{\pi}{3}}{2(5\sqrt{13})}.\frac{\pi}{90} = \frac{\pi}{\sqrt{39}}$.

5.10.2 Evaluations of Extreme-Values and Intervals of Monotone

Extreme-Values: In mathematical term, the word '**Extreme**' means either the smallest or the largest on a given scale of measurement. It is also used to mean the first and the last terms in a given proportion. So, in function analysis, it is a collective name used to mean either minimum or maximum values of a function. Furthermore, depending on the extent of the domain considered, there are two types of extreme values: Absolute extreme- value and Relative Extreme Values.

I) Absolute (Global) Extreme Values: Compared on the entire domain

Let a and b be points in the domain D of a function f. Then, the value $f(a)$ is said to be absolute *maximum value* of f on D if and only if $f(x) \le f(a)$ for all x in D and the value $f(b)$ is said to be absolute *minimum value* of f on D if and only if $f(b) \le f(x)$ for all x in D. The absolute maximum and minimum values of f are called *absolute extreme values*.

II) Relative (Local) Extreme-Values: On small subset of the domain

Let a and b be points in the domain D of a function f. Then, the value $f(a)$ is said to be *local maximum value* of f if there is an open interval I containing a and contained in D such that $f(x) \le f(a)$ for all x in I and $f(b)$ is said to be *local minimum value* of f if there is an open interval I containing b and contained in D so that $f(b) \le f(x)$ for all x in I. The relative maximum and relative minimum values of f are called *relative extreme values*.

III) Intervals of Monotone:

i) A function f is said to be *increasing* on an interval if and only if for any two numbers x and y in the interval $x < y$ implies $f(x) \le f(y)$.

ii) A function f is said to be *decreasing* on an interval if and only if for any two numbers x and y in the interval $x < y$ implies $f(x) \ge f(y)$.

iii) A function which is either increasing or decreasing is called *monotone*.

The interval where f is increasing or decreasing is said to be *interval of monotone*.

Critical Numbers and Extreme-Value Theorem:

A number c in the domain of f is said to be critical number of f if and only if either $f'(c)=0$ or $f'(c)$ does not exist.

i) For Absolute Extreme Values: If f is continuous on a closed interval $[a,b]$, then f has both maximum and minimum values on $[a,b]$.

ii) Extreme-Value Theorem for Relative Extreme-Values:

If a function f has extreme values on an open interval (a,b), then these values occur only at the critical numbers of f.

A) Evaluation of Absolute Extreme-Values

Procedures to find Absolute Extreme Values on closed Intervals:

Step-1: Find all the critical numbers of f that lie in the given interval only.

Step-2: Evaluate f at all the critical numbers and at the endpoints a and b.

Decision: The *largest* is maximum value and the *smallest* is minimum value.

Examples:

1. For each of the following functions,
   i) First find all the critical numbers in the whole domain of the function
   ii) Give the absolute maximum and minimum values on the given interval

a) $f(x)=x^3-12x+9$ on[-3,3] b) $f(x)=x^3-3x^2-9x$ on[-2,2]

c) $f(x)=x^2-3x^{\frac{2}{3}}-2$ on$[-8,8]$ d) $f(x)=8x\sqrt{1-x^2}$ on[-1,1]

e) $f(x)=x^2e^{-2x}$ on [-2,1] f) $f(x)=|x^3-9x|$ on[1,4]

g) $f(x)=x^4-8x^2+9$ on[-1,1]

Solution:

a) Here, $f'(x)=3x^2-12=0\Rightarrow x^2-4=0\Rightarrow x^2=4\Rightarrow x=-2, x=2$.

Since the domain of f is the set of all real numbers, both $x=-2, x=2$ are the critical numbers of f. Besides, both of them are found on the given interval.

So, evaluate f both at $x=-2, x=2$ and at the end points $x=-3, x=3$.

That is $f(-2)=25$, $f(2)=-7$, $f(-3)=18$, $f(3)=0$.

Therefore, on the given interval [-3,3], the maximum value is $f(-2)=25$ and the minimum value is $f(2)=-7$.

b) $f'(x)=3x^2-6x-9=0 \Rightarrow x^2-2x-3=0 \Rightarrow (x+1)(x-3)=0 \Rightarrow x=-1,\ x=3$

Since the domain of f is the set of all real numbers, both $x=-1, x=3$ are the critical numbers of f. But from these only $x=-1$ is found on the given interval. So we will evaluate f only at $x=-1$ and at the end points $x=-2, x=2$.

That is $f(-1)=5,\ f(-2)=-2,\ f(2)=-22$. Hence, on the given interval $[-2,2]$, the maximum value is 5 and the minimum value is -22.

c*) $f'(x)=2x-2x^{-\frac{1}{3}}=0 \Rightarrow 2x=\frac{2}{x^{\frac{1}{3}}} \Rightarrow x^{\frac{4}{3}}=1 \Rightarrow x^2=1 \Rightarrow x=\pm 1$

$f(x)=x^2-3x^{\frac{2}{3}}-2 \quad \text{on}[-8,8]$

Since the domain of f is the set of all real numbers, both $x=-1, x=1$ are the critical numbers of f and found on the given interval.

Besides, $f'(x)$ does not exist when $x^{\frac{1}{3}}=0 \Rightarrow x=0$.

Hence, we have got three critical numbers $x=0, x=-1, x=1$.

So, evaluate f at three critical numbers and at the end points $x=-8, x=8$.

That is $f(-1)=f(1)=-4,\ f(0)=-2, f(-8)=f(8)=50$.

Hence, on the interval $[-8,8]$, the maximum is 50 and the minimum is -4.

d) Here, $f'(x)=0 \Rightarrow \frac{8-16x^2}{\sqrt{1-x^2}}=0 \Rightarrow 8-16x^2=0 \Rightarrow x=\pm\frac{1}{\sqrt{2}}$.

Again, $f'(x)$ does not exist implies that $1-x^2=0 \Rightarrow x=1, x=-1$.

Hence, the critical numbers are $\pm 1/\sqrt{2}, \pm 1$.

Besides, $f(\frac{1}{\sqrt{2}})=4,\ f(-\frac{1}{\sqrt{2}})=-4,\ f(-1)=f(1)=0$.

Therefore, the maximum is 4 and the minimum is 0.

e) Here, $f'(x)=2xe^{-2x}-2x^2e^{-2x}=2x(1-x)e^{-2x}$. Then, by equating the derivative to zero, the critical numbers are $f'(x)=0 \Rightarrow x=0,\ x=1$.

Besides, $f(0)=0,\ f(-2)=4e^4,\ f(1)=\frac{1}{e^2}$.

Hence, the maximum value is $4e^4$ and the minimum value is 0.

f) Observe that always when you are asked to find the derivative of a function in absolute value $f(x)=|g(x)|$, use the relation $f(x)=|g(x)|=\sqrt{[g(x)]^2}$ such that the derivative becomes $f'(x)=\frac{g(x)g'(x)}{\sqrt{[g(x)]^2}}$.

Thus, $f(x) = \dfrac{(x^3-9x)(3x^2-9)}{\sqrt{(x^3-9x)^2}}$. Then, $f'(x)=0 \Rightarrow x=\sqrt{3},\ x=-\sqrt{3}$.

Besides, $f'(x)$ does not exist at $x=-3, x=0, x=3$. Since the domain of the function is the set of all real numbers, all of the numbers $-3, -\sqrt{3}, 0, \sqrt{3}$ and 3 are critical numbers. But among these, only $x=\sqrt{3}, x=3$ are found on the given interval. So, $f(1)=8,\ f(\sqrt{3})=6\sqrt{3},\ f(3)=0,\ f(4)=28$.

Hence, the maximum value is 28 and the minimum value is 0.

g) $f'(x)=0 \Rightarrow 4x^3-16x=0 \Rightarrow 4x(x^2-4)=0 \Rightarrow x=0,\pm 2$. From these, only $x=0$ is found in [-1,1]. Besides, $f(0)=9,\ f(-1)=f(1)=2$.

So, the maximum value is 9 and the minimum value is 0.

2*. Find the absolute maximum and minimum of

a) $f(x)=\begin{cases} x^3-12x; 0\le x\le 3 \\ x^2+2x+1; 3<x\le 4 \end{cases}$ on theinterval [0,4]

b) $f(x)=\begin{cases} 7x^2+14x+1; -1\le x\le 0 \\ x^2-4x+1; 0<x\le 3 \end{cases}$ on theinterval [-1,3]

Solution: In each case, first let's find the critical numbers.

a) $f'(x)=\begin{cases} 3x^2-12; 0\le x\le 3 \\ 2x+2; 3<x\le 4 \end{cases} \Rightarrow f'(x)=0 \Rightarrow x=\pm 2, x=-1$.

Besides, $f'(x)$ does not exist at $x=3$ because it is not continuous at 3.

Hence, the critical numbers are $x=\pm 2, x=-1, x=3$.

Of these critical numbers only $x=2$ and $x=3$ are found in the given interval.

Thus, $f(0)=0,\ f(2)=-16, f(3)=-9, f(4)=25$.

Therefore, the absolute maximum value is $f(4)=25$ and the absolute minimum value is $f(2)=-16$.

b) $f'(x)=\begin{cases} 14x+14; -1\le x\le 0 \\ 2x-4; 0<x\le 3 \end{cases}$

Then, $f'(x)=0 \Rightarrow 14x+14=0, 2x-4=0 \Rightarrow x=-1,2$.

Of these critical numbers only $x=-1$ is found in the given interval.

Again, at the boundary (separating point) $x=0$, $f'(x)$ does not exist.

So, the critical numbers of the function are $x=-1,0$ and $x=2$.

Besides, $f(-1)=-6, f(0)=1, f(2)=-3, f(3)=-2$. Therefore, the absolute maximum value is $f(0)=1$ and the absolute minimum value is $f(-1)=-6$.

3. Find the absolute maximum and minimum values of the following functions.

a) $f(x)=2\sin x+\sin 2x$ on $[0,2\pi]$ b) $f(x)=\sin x-\cos x$ on $[0,\pi]$

c) $f(x)=2\sin x-\cos 2x+5$ on $[0,2\pi]$ d*) $f(x)=5-\cos \pi x$ on[-2,3]

Solution:

a) $f'(x)=2\cos x+2\cos 2x=0 \Rightarrow 2\cos^2 x+2\cos x-1=0$

$$\Rightarrow (2\cos x-1)(\cos x+1)=0 \Rightarrow \cos x=1/2, \cos x=-1$$

$$\Rightarrow x=\pi/3, \pi, 5\pi/3 \text{ in } [0,2\pi]$$

So, the absolute maximum is $\frac{3\sqrt{3}}{2}$ and the absolute minimum is $-\frac{3\sqrt{3}}{2}$

b) $f'(x)=\cos x+\sin x=0 \Rightarrow \tan x=-1$. So, on $[0,\pi]$, $x=\frac{3\pi}{4}$.

Besides, $f(0)=-1, f(\frac{3\pi}{4})=\sqrt{2}, f(\pi)=1$.

Hence, the absolute maximum is $\sqrt{2}$ and the absolute minimum is -1.

c) $f'(x)=0 \Rightarrow 2\cos x(1+2\sin x)=0 \Rightarrow x=\cos^{-1}(0), x=\sin^{-1}(-\frac{1}{2})$.

Therefore, the critical numbers on $[0,2\pi]$ are $x=\frac{\pi}{2}, x=\frac{3\pi}{2}, x=\frac{7\pi}{6}, x=\frac{11\pi}{6}$

Thus, $f(\frac{\pi}{2})=8,\ f(\frac{3\pi}{2})=4, f(\frac{7\pi}{6})=\frac{7}{2}, f(\frac{11\pi}{6})=\frac{7}{2}, f(0)=4,\ f(2\pi)=4$

Therefore, the maximum value is 8 and the minimum value is $7/2$.

B) Evaluation of Relative Extreme Value and Interval of Monotone

Suppose f is continuous and differentiable on an open interval I containing arbitrary critical number c of f (but f may or may not be differentiable at c). Then, we have the following FIRST DERIVATIVE TESTS for relative extreme values and intervals of monotone.

I) First Derivative Test for Relative Extreme Values

a) If the sign of f' (in going from left to right) changes from positive to negative at c, then f has a relative maximum value at c.

b) If the sign of f' (in going from left to right) changes from negative to positive at c, then f has a relative minimum value at c.

c) If the sign of f' does not change at c, then f has no relative maximum or relative minimum value at c.

II) First Derivative Test for Intervals of Monotone

a) If $f'(x) > 0$ for all x in the interior of I, then f is increasing on I.

b) If $f'(x) < 0$ for all x in the interior of I, then f is decreasing on I.

c) If $f'(x) = 0$ for all x in the interior of I, then f is constant on I.

The theorem remains true even when $f'(x) = 0$ for finitely many points in I.

Summarized Procedures:

To evaluate relative extreme values and intervals of monotone, please bear in mind the following steps:

Procedures to find Relative extreme values:

Step-1: Find all the critical numbers of f.

Step-2: Form the sign chart for $f'(x)$ using the critical numbers as poles.

Step-3: Analyze the sign using FDT to determine the required values.

Examples:

1. Using the above procedures and First Derivative Tests, find
   i) Find the relative maximum and minimum values for each function.
   ii) Determine the intervals of increasing and decreasing fot each function.

a) $f(x) = 2x^3 + 9x^2 - 4$ b) $f(x) = 8 + 5x^3 - 3x^5$

c) $f(x) = 3x + \frac{12}{x}$ d) $f(x) = \frac{3 - x^2}{e^x}$

e) $f(x) = x^4 - 8x^2 + 6$ f) $f(x) = x^3 - 2x^2 - 4x + 4$

g) $f(x) = \frac{3x^2}{x+2}$ h) $f(x) = \frac{2x^2 - 1}{x^4}$

i) $f(x) = x^4 - 4x^3 + 4x^2 - 9$ j) $f(x) = 8 + 5x^4 - x^5$

Solution: Use the above procedures for clarity.

a) **Step-1:** Find all the critical numbers of f

That is $f'(x) = 6x^2 + 18x = 0 \Rightarrow 6x(x+3) = 0 \Rightarrow x = 0, x = -3$.

Step-2: Form the sign chart for $f'(x)$ using the critical numbers as poles

Interval	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < \infty$
$6x$	$---$	$---$	$+++$
$x+3$	$---$	$+++$	$+++$
$f'(x) = 6x(x+3)$	$+++$	$---$	$+++$

Step-3: Analyze the sign using FDT to determine the required values

For Relative Extreme values: Here, the sign of $f'(x)$ changes from positive to negative at $x = -3$ and from negative to positive at $x = 0$.

This means that by FDT, f has relative maximum value at $x = -3$ and relative minimum value at $x = 0$. Therefore, the relative maximum value is $f(-3) = 23$ and the relative minimum value is $f(0) = -4$.

For Intervals of Monotones: By FDT, a function is increasing on an interval where the last row of the sign chart shows positive sign and decreasing on an interval where the last row of the sign chart shows negative sign.

Therefore, by looking the above sign chart, we have

Interval of Increasing: The function is increasing on $(-\infty,-3]\cup[-3,\infty)$

Interval of Decreasing: The function is decreasing on $[-3,0]$

b) **Step-1:** Find all the critical numbers of f

That is $f'(x)=0 \Rightarrow f'(x)=15x^2-15x^4=0 \Rightarrow 15x^2(1-x^2)=0$

$$\Rightarrow 15x^2(1-x)(1+x)=0 \Rightarrow x=0, x=\pm1$$

Step-2: Form the sign chart for $f'(x)$ using the critical numbers as poles

Intervals	$x<-1$	$-1\le x\le 0$	$0\le x\le 1$	$x>1$
$15x^2$	+++	+++	+++	+++
$1-x$	+++	+++	+++	−−−
$1+x$	−−−	+++	+++	+++
$f'(x)=15x^2(1-x)(1+x)$	−−−	+++	+++	−−−

Step-3: Analyze the sign using FDT to determine the required values

For Relative Extreme values: Here, the sign of $f'(x)$ changes from positive to negative at $x=1$. Therefore, by FDT, f has relative maximum value at $x=1$.

Relative Maximum: The relative maximum value is $f(1)=10$

At the critical number $x=0$, the sign of $f'(x)$ does not change. That means it remains positive on both sides. Therefore, by FDT, there is no relative extreme value at $x=0$. This means f has no relative maximum or minimum at $x=0$.

Besides, the sign of $f'(x)$ changes from negative to positive at $x=-1$.

Therefore, by FDT, f has relative minimum value at $x=-1$.

Relative Minimum: The relative minimum value is $f(-1)=6$

For Intervals of Monotones: By FDT, a function is increasing on an interval where the last row of the sign chart shows positive sign and decreasing on an interval where the last row of the sign chart shows negative sign.

Therefore, by looking the above sign chart, we have

Interval of Increasing: The function is increasing on $[-1,1]$

Interval of Decreasing: The function is decreasing on $(-\infty,-1]\cup[1,\infty)$

c) **Step-1:** Find all the critical numbers of f

Here, $f'(x)=3-\frac{12}{x^2}=\frac{3x^2-12}{x^2}=\frac{3(x^2-4)}{x^2}=\frac{3(x-2)(x+2)}{x^2}$.

Critical numbers: $f'(x)=0 \Rightarrow 3(x-2)(x+2)=0 \Rightarrow x=2, x=-2$.

Step-2: Form the sign chart for $f'(x)$ using the critical numbers as poles. Besides, since f is undefined at 0, we must use $x=0$ as a pole in the construction of the sign chart.

Intervals	$x<-2$	$-2\le x<0$	$0<x\le 2$	$x>2$
$x-2$	---	---	---	+++
$x+2$	---	+++	+++	+++
$f'(x)=\frac{3(x-2)(x+2)}{x^2}$	+++	---	---	+++

Step-3: Analyze the sign using FDT to determine the required values

For Relative Extreme values: Here, the sign of $f'(x)$ changes from positive to negative at $x=-2$ and from negative to positive at $x=2$. Therefore, by FDT, f has relative maximum value at $x=-2$ and relative minimum value at $x=2$.

Relative Maximum: The relative maximum value is $f(-2)=-6-6=-12$.

Relative Minimum: The relative minimum value is $f(2)=6+6=12$.

For Intervals of Monotones: By using FDT for interval of monotone;

Interval of Increasing: The function is increasing on $(-\infty,-2]\cup[2,\infty)$

Interval of Decreasing: The function is decreasing on $[-2,0)\cup(0,2]$

e) **Step-1:** Find all the critical numbers of f

That is $f'(x)=4x^3-16x=0 \Rightarrow 4x(x-2)(x+2)=0$.

Step-2: Form the sign chart for $f'(x)$ using the critical numbers as poles

Intervals	$x<-2$	$-2\le x\le 0$	$0\le x\le 2$	$x>2$
$4x$	---	---	+++	+++
$x-2$	---	---	---	+++
$x+2$	---	+++	+++	+++
$f'(x)=4x(x-2)(x+2)$	---	+++	---	+++

Step-3: Analyze the sign using FDT to determine the required values

For Relative Extreme values: Here, the sign of $f'(x)$ changes from positive to negative at $x=0$ and from negative to positive at $x=-2$ and $x=2$.

Therefore, by FDT, f has relative maximum value at $x=0$ and relative minimum value at $x=-2$ and $x=2$.

Relative Maximum: The relative maximum value is $f(0)=6$

Relative Minimum: The relative minimum value is $f(-2)=f(2)=-10$

For Intervals of Monotones: By FDT, a function is increasing on an interval where the last row of the sign chart shows positive sign and decreasing on an interval where the last row of the sign chart shows negative sign.

Therefore, by looking the above sign chart, we have

Interval of Increasing: The function is increasing on $(-\sqrt{2},0]\cup[\sqrt{2},\infty)$

Interval of Decreasing: The function is decreasing on $(-\infty,-\sqrt{2}]\cup[0,\sqrt{2}]$

f) **Step-1:** Find all the critical numbers of f

That is $f'(x)=3x^2-4x-4=0 \Rightarrow (x-2)(3x+2)=0 \Rightarrow x=2, x=-\frac{2}{3}$.

Step-2: Form the sign chart for $f'(x)$ using the critical numbers as poles

Now construct the sign chart using these critical numbers.

Interval	$-\infty<x<-2/3$	$-2/3<x<2$	$2<x<\infty$
$x-2$	− − −	− − −	+ + +
$x+2/3$	− − −	+ + +	+ + +
$f'(x)=(x-2)(3x+2)$	+ + +	− − −	+ + +

Step-3: Analyze the sign using FDT to determine the required values

For Relative Extreme values: Here, the sign of $f'(x)$ changes from positive to negative at $x=-\frac{2}{3}$ and from negative to positive at $x=2$. Therefore, by FDT, f has relative maximum value at $x=-\frac{2}{3}$ and relative minimum value at $x=2$.

Relative Maximum: The relative maximum value is $f(-\frac{2}{3})=\frac{20}{27}$

Relative Minimum: The relative minimum value is $f(2)=-4$

For Intervals of Monotones: By FDT, a function is increasing on an interval where the last row of the sign chart shows positive sign and decreasing on an interval where the last row of the sign chart shows negative sign.

Therefore, by looking the above sign chart, we have

Interval of Increasing: The function is increasing on $(-\infty,-\frac{2}{3}]\cup[2,\infty)$

Interval of Decreasing: The function is decreasing on $[-\frac{2}{3},2]$

g) Here $f'(x)=\dfrac{6x(x+2)-3x^2}{(x+2)^2}=\dfrac{3x(x+4)}{(x+2)^2}$.

Hence, the critical numbers are $x=0,\ x=-4$.

Interval	$-\infty<x<-4$	$-4<x<0$	$0<x<\infty$
$3x$	---	---	+++
$x+4$	---	+++	+++
$f'(x)=\dfrac{3x(x+4)}{(x+2)^2}$	+++	---	+++

As we see from the sign chart, $f'(x)$ changes sign from negative to positive at $x=0$ and thus f has relative minimum and the relative minimum is $f(0)=0$. On the other hand, $f'(x)$ changes sign from positive to negative at $x=-4$ and thus f has relative maximum value $f(-4)=-24$.

2. Find the relative maximum and minimum values.

a) $f(x)=x^3e^x$ b) $f(x)=x^4e^{\frac{-x^2}{2}}$ c) $f(x)=x^x$

d) $f(x)=(x+3)^2(x-1)^2$ f) $f(x)=(3x-x^3)^{\frac{2}{3}}$ e) $f(x)=x^2e^{-x}$

g) $f(x)=x^3-3x+1$ h) $f(x)=x^{\frac{1}{x}}$ i) $f(x)=x^2e^x$

Solution:

a) Here $f'(x)=3x^2e^x+x^3e^x=x^2(x+3)e^x$. Hence, the critical numbers are $x=0,\ x=-3$. Now let's construct the sign chart for $f'(x)$.

Interval	$-\infty < x < -3$	$-3 < x < 0$	$0 < x < \infty$
x^2	+++	+++	+++
$x+3$	−−−	+++	+++
$f'(x) = x^2(x+3)e^x$	−−−	+++	+++

Here, $f'(x)$ changes sign from negative to positive at $x=-3$ and thus f has relative minimum value at this critical number.

The relative minimum value is $f(-3) = -27e^{-3}$. On the other hand, $f'(x)$ does not change sign at $x=0$. Thus f has no extreme value at this critical number.

b) Here, $f'(x) = 4x^3e^{-\frac{x^2}{2}} - x^5e^{-\frac{x^2}{2}} = x^3(4-x^2)e^{-\frac{x^2}{2}} = x^3(2-x)(2+x)e^{-\frac{x^2}{2}}$

Critical numbers: $f'(x) = 0 \Rightarrow x^3(2-x)(2+x)e^{-\frac{x^2}{2}} = 0 \Rightarrow x = 0, 2, -2$.

Now let's construct the sign chart for $f'(x)$.

Interval	$-\infty < x < -2$	$-2 < x < 0$	$0 < x < 2$	$2 < x < \infty$
x^3	−−−	−−−	+++	+++
$2-x$	+++	+++	+++	−−−
$x+2$	−−−	+++	+++	+++
$f'(x) = x^3(2-x)(x+2)e^{-\frac{x^2}{2}}$	+++	−−−	+++	−−−

As we see from the sign chart, $f'(x)$ changes sign from negative to positive at $x=0$ and thus f has relative minimum value at this critical number.

The relative minimum value is $f(0)=0$. On the other hand, $f'(x)$ changes sign from positive to negative at $x=-2, x=2$ and thus f has relative maximum value at these critical numbers. The relative maximumis $f(-2) = f(2) = 16e^{-2}$.

c) Here, $f'(x) = x^x(1+\ln x) = 0 \Rightarrow x = \frac{1}{e}$. Besides, $f'(x) < 0$ for $0 < x < \frac{1}{e}$ and $f'(x) > 0$ for $x > \frac{1}{e}$. Then, $f(\frac{1}{e}) = (\frac{1}{e})^{\frac{1}{e}}$ is relative minimum value.

d) Here, $f'(x) = 2(x+3)(x-1)^2 + 2(x+3)^2(x-1) = 4(x+3)(x-1)(x+1)$.

Interval	$-\infty < x < -3$	$-3 < x < -1$	$-1 < x < 1$	$1 < x < \infty$
$x+3$	---	+++	+++	+++
$x+1$	---	---	+++	+++
$x-1$	---	---	---	+++
$f'(x) = 4(x+3)(x-1)(x+1)$	---	+++	---	+++

Therefore, it is increasing on $[-3,-1] \cup [1,\infty)$ and decreasing on $(-\infty,-3] \cup [-1, 1]$.

e) Here, $f'(x) = \frac{2}{3}(\frac{3-3x^2}{(3x-x^3)^{\frac{1}{3}}}) = \frac{2(1-x)(x+1)}{\sqrt[3]{x(\sqrt{3}-x)(x+\sqrt{3})}}$.

Interval	$-\infty < x < -\sqrt{3}$	$-\sqrt{3} < x < -1$	$-1 < x < 0$	$0 < x < 1$	$1 < x < \sqrt{3}$	$\sqrt{3} < x < \infty$
$x+1$	---	---	+++	+++	+++	+++
$1-x$	+++	+++	+++	+++	---	---
x	---	---	---	+++	+++	+++
$\sqrt{3}-x$	+++	+++	+++	+++	+++	---
$x+\sqrt{3}$	---	+++	+++	+++	+++	+++
$f'(x) = \frac{2(1-x)(x+1)}{\sqrt[3]{x(\sqrt{3}-x)(x+\sqrt{3})}}$	---	+++	---	+++	---	+++

Therefore, the function is increasing on $[-\sqrt{3},-1] \cup [0,1] \cup [\sqrt{3},\infty)$ and decreasing on $(-\infty,-\sqrt{3}] \cup [-1,0] \cup [1,\sqrt{3}]$.

f) Here, $f'(x) = 2xe^{-x} - x^2e^{-x} = x(2-x)e^{-x}$. Now let's construct the sign chart.

Interval	$-\infty < x < 0$	$0 < x < 2$	$2 < x < \infty$
x	---	+++	+++
$2-x$	+++	+++	---
e^{-x}	+++	+++	+++
$f'(x) = x(2-x)e^{-x}$	---	+++	---

Therefore, the function is increasing on $[0, 2]$ and decreasing on $(-\infty,0] \cup [2,\infty)$.

g) $f'(x)=3x^2-3=3(x-1)(x+1)$. Now let's construct the sign chart for $f'(x)$.

Interval	$-\infty<x<-1$	$-1<x<1$	$1<x<\infty$
$x-1$	$---$	$---$	$+++$
$x+1$	$---$	$+++$	$+++$
$f'(x)=3(x-1)(x+1)$	$+++$	$---$	$+++$

Since $f'(x)>0$ on $-\infty<x<-1$, $1<x<\infty$ and $f'(x)<0$ on $-1<x<1$.

Therefore, the function is increasing on $(-\infty,-1]\cup[1,\infty)$ and decreasing on $[-1,1]$.

Theorem: Suppose c is an interior point in the domain of f. If f has a relative extreme value at c and $f'(c)$ exists, then $f'(c)=0$. This Theorem tells us that the first derivative of a function is always zero at an interior point where the function has a relative extreme if the derivative exists.

Examples: In each of the following problems, find the values of a, b, c..

a) If $f(x)=x^2+\frac{a}{x}$ has a relative extreme value at $x=3$.

b) If $f(x)=ax^3+bx-5$ has a relative minimum value of -6 at $x=1/2$.

c) If $g(2)=1$ is a relative maximum value of $g(x)=axe^{bx^2}$.

Solution:

a) Here, $f(x)=x^2+\frac{a}{x}\Rightarrow f'(x)=2x-\frac{a}{x^2}$. Since f has a relative extreme value at $x=3$, we have $f'(3)=0\Rightarrow 6-\frac{a}{9}=0\Rightarrow a=54$.

b) $f'(x)=3ax^2+b$. Since f has a relative minimum value at $x=1/2$, we have $f'(1/2)=0\Rightarrow 3a/4+b=0\Rightarrow b=-3a/4$. Then, using $b=-3a/4$, we get $\frac{a}{8}-\frac{3a}{8}-5=-6\Rightarrow -\frac{a}{4}=-1\Rightarrow a=4$ and $b=-3$.

III) The Second Derivative Test for Relative Extreme Values

Sometimes, we may need to decide about relative extreme values at critical points quickly without building the sign chart for the first derivative. This is possible by applying the second derivative test which is stated as follow.

Theorem (Second Derivative Test, SDT): Suppose f is continuous on an interval I containing c such that $f'(c)=0$ and $f''(c)$ exists. Then,

a) If $f''(c)<0$, then f has a local maximum value at c.

b) If $f''(c)>0$, then f has a local minimum value at c.

c) If $f''(c)=0$, then the test fails. In such case, we apply first derivative test.

Examples:

1. Using SDT, find the relative maximum and relative minimum values.

$a)\ f(x)=3x^5-20x^3 \qquad b)\ f(x)=x+\frac{9}{x} \qquad c)\ f(x)=x^4-4x^2+11$

$d)\ f(x)=x^3-12x^2+45x+4 \qquad e)\ f(x)=\frac{\sqrt{x^2+7}}{3}+\frac{5-x}{4}$

Solution:

a) $f'(x)=15x^4-60x^2=15x^2(x^2-4)$. So, the critical numbers are $x=0,2,-2$.
Now, let's test for relative extreme values using SDT on $f''(x)=60x^3-120x$.
Since $f''(2)=240>0$, by SDT f has relative minimum at $x=2$ and the relative minimum value is $f(2)=-64$. Since $f''(-2)=-240<0$, by SDT f has relative maximum at $x=-2$ and the relative maximum value is $f(-2)=64$. However, $f''(0)=0$ which means that SDT is not applicable (it fails) to determine whether relative maximum or minimum occurs at $x=0$. By applying FDT, we see that f' does not change sign at 0 and thus f has no extreme values at $x=0$.

b) Since $f'(x)=1-\frac{9}{x^2}$, the critical numbers are $x=-3,3$. Now, let's test for relative extreme values. Here, $f''(x)=\frac{18}{x^3}$. Since, $f''(3)=\frac{2}{3}>0$, by SDT, $f(3)=6$ is a relative minimum value.

Since $f''(-3) = -\frac{2}{3} < 0$; $f(-3) = -6$ is a relative maximum.

c) $f'(x) = 4x^3 - 8x$ .Then, $f'(x) = 0 \Rightarrow 4x^3 - 8x = 0 \Rightarrow x = 0, x = \sqrt{2}, x = -\sqrt{2}$.

Besides, $f''(x) = 12x^2 - 8 \Rightarrow f''(0) = -8,\ f''(\sqrt{2}) = 16,\ f''(-\sqrt{2}) = 16$.

Since $f''(0) < 0$ by second derivative test f has a relative maximum value at $x = 0$ and the relative maximum value is $f(0) = 11$. On the other hand, since $f''(\sqrt{2}) > 0$ and $f''(-\sqrt{2}) > 0$, $f(\pm\sqrt{2}) = 7$ is a relative minimum value.

d) $f'(x) = 3x^2 - 24x + 45 = 3(x-3)(x-5)$. Hence, the critical numbers are $x = 3,5$. Now, let's test for relative extreme values using SDT on $f''(x) = 6x - 24$. Since, $f''(3) = -6 < 0$,by SDT f has relative maximum at $x = 3$ and the relative maximum value is $f(3) = 50$. Since $f''(5) = 6 > 0$,by SDT f has relative minimum at $x = 5$ and the minimum value is $f(5) = 46$.

e) $f'(x) = 0 \Rightarrow \frac{x}{3\sqrt{x^2+7}} - \frac{1}{4} = 0 \Rightarrow 9(x^2+7) = 16x^2 \Rightarrow x = -3,\ x = 3$

Therefore, by SDT, f has a relative minimum at $x = 3, x = -3$ and the relative minimum values are $f(3) = \frac{11}{3}$ and $f(-3) = \frac{10}{3}$.

2. The total cost of ordering and transporting x units of a certain quantity is given by $C(x) = 3x + \frac{7500}{x}$ birr. How many units must be ordered so as to make the cost minimum.

Solution: To minimize the cost, simply find its critical number.

That is $C'(x) = 3 - \frac{7500}{x^2} = 0 \Rightarrow \frac{7500}{x^2} = 3 \Rightarrow x^2 = 2500 \Rightarrow x = \pm 50$

Here, only $x = 50$ is the valid critical number. Besides, by second derivative test, we have $C''(x) = \frac{15000}{x^3} \Rightarrow C''(50) = \frac{15000}{(50)^3} = \frac{3}{25} > 0$.

That means the cost function is minimum when $x = 50$ units are ordered.

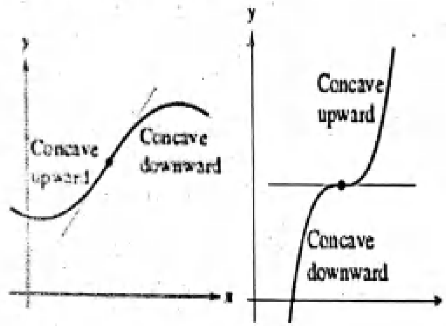
5.10.3 Evaluation of Intervals of Concavity and Inflection Point

Definition: Let f be differentiable on an open interval I. Then, the graph of f is said to be

a) *concave up* on I if f' is *increasing* on I.

b) *concave down* on I if f' is *decreasing* on I.

Geometrically, on the interval where the graph of a function is concave up, its graph lies below its tangent line and on the interval where the graph of a function is concave down, its graph lies above its tangent line as shown in the diagram.



Concavity of graph of a function

Second Derivative Test for concavity: Suppose f is twice differentiable on an open interval I Then, the graph of

a) f is *concave up* on I if $f''(x) > 0$ for all x in I.

b) f is *concave down* on I if $f''(x) < 0$ for all x in I.

The interval where the graph of f is concave up or concave down is called *interval of concavity*.

Definition: A point on the graph of f is said to be an *inflection point* if and only if the concavity of f changes either from *concave up* to *concave down* or from *concave down* to *concave up* at that point. Since an inflection point is a point where the graph of f changes its concavity, f'' changes its sign at such point. Thus, we have the following test for an inflection point.

Theorem (Test for Inflection point): Suppose f is continuous and f'' exists on an open interval containing $x = c$. Then, the point $(c, f(c))$ is an inflection point of f if f'' has opposite sign on different sides of c.

Procedures to find interval of concavity and inflection points:

Step-1: *Find all the points where $f''(x)=0$ or $f''(x)$ does not exist.*

Step-2: *Form the sign chart for $f''(x)$ using the points obtained in step 1.*

Step-3: *Apply SDT for concavity and decide.*

Examples: Find the interval of concavity and inflection points.

c) $f(x)=x^4-6x^2+9$ b) $f(x)=1+30x^3-x^5$ c) $f(x)=x^4e^x$

d) $f(x)=\dfrac{4x}{x^2+1}$ e) $f(x)=\dfrac{x^4}{6}-2x^2+4$ f) $f(x)=\dfrac{x^4}{6}-2x^2+4$

Solution:

a) For clarity, let's follow the above steps:

Step-1: Find all x where $f''(x)=0$ or $f''(x)$ does not exist.

Here, $f''(x)=12x^2-12=12(x-1)(x+1)=0 \Rightarrow x=1, x=-1$

Step-2: Construct the sign chart for $f''(x)$ using these numbers as poles.

Intervals	$x<-1$	$-1\le x\le 1$	$x>1$
$x-1$	$---$	$---$	$+++$
$x+1$	$---$	$+++$	$+++$
$f'(x)=12(x-1)(x+1)$	$+++$	$---$	$+++$

Step-3: Analysis using SDT for concavity:

By SDT, the graph of a function f is concave up on an interval where $f''(x)>0$ and concave down on an interval where $f''(x)<0$.

Therefore, using this test on the sign chart, the intervals of concavity are as follow:

Interval of concave up: The graph is concave u on $(-\infty,-1)\cup(1,\infty)$

Interval of concave down: The graph is concave down on $(-1,1$

Inflection point: Since the concavity changes at $x=-1$ and $x=1$, the inflection points are $(-1, f(-1)=(-1,4)$ & $(1, f(1)=(1,4)$.

b) **Step-1:** Find all x where $f''(x)=0$ or $f''(x)$ does not exist.

Here, $f''(x)=180x-20x^3=20x(9-x^2)=20x(3-x)(3+x)$

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Step-2: Construct the sign chart for $f''(x)$ using these numbers as poles.

Now construct the sign chart using the points $x=-3, x=0, x=3$.

Interval	$-\infty<x<-3$	$-3<x<0$	$0<x<3$	$3<x<\infty$
$20x$	---	---	+++	+++
$x+3$	---	+++	+++	+++
$3-x$	+++	+++	+++	---
$f''(x)=20x(x+3)(3-x)$	+++	---	+++	---

Step-3: Analysis using SDT for concavity: From the sign chart,by SDT;

Interval of concave up: The graph is concave u on $(-\infty,-3)\cup(0,3)$

Interval of concave down: The graph is concave down on $(-3,0)\cup(3,\infty)$

Inflection points: The inflection points are $(-3,f(-3)),\ (0,f(0)),\ (3,f(3))$.

c) Here, $f''(x)=12x^2e^x+8x^3e^x+x^4e^x=x^2e^x(x+6)(x+2)$.

Now construct the sign chart using the points $x=-6, x=-2, x=0$.

Interval	$-\infty<x<-6$	$-6<x<-2$	$-2<x<0$	$0<x<\infty$
x^2	+++	+++	+++	+++
$x+6$	---	+++	+++	+++
$x+2$	---	---	+++	+++
$f''(x)=x^2(x+6)(x+2)e^x$	+++	---	+++	+++

Therefore, the function is concave up on $(-\infty,-6)\cup(-2,0)\cup(0,\infty)$ and concave down on $(-6,-2)$. Besides, the concavity of the graph of f changes at $x=-6, x=-2$ and the inflection points are $(-6,f(-6)),\ (-2,f(-2))$.

Notice that the function has no inflection point at zero. (Do you see why?)

$d)\ f'(x)=\dfrac{4(1-x^2)}{(x^2+1)^2}\Rightarrow f''(x)=\dfrac{8x(x^2-3)}{(x^2+1)^3}=0\Rightarrow x=0, x=\sqrt{3}, x=-\sqrt{3}$.

Now construct the sign chart using these critical points as poles.

Interval	$-\infty<x<-\sqrt{3}$	$-\sqrt{3}<x<0$	$0<x<\sqrt{3}$	$\sqrt{3}<x<\infty$
$8x$	---	---	+++	+++
$x-\sqrt{3}$	---	---	---	+++
$x+\sqrt{3}$	---	+++	+++	+++
$(x^2+1)^3$	+++	+++	+++	+++
$f''(x)=\frac{8x(x-\sqrt{3})(x+\sqrt{3})}{(x^2+1)^3}$	---	+++	---	+++

Therefore, it is concave up on $(-\sqrt{3},0)\cup(\sqrt{3},\infty)$ and concave down on $(-\infty,-\sqrt{3})\cup(0,\sqrt{3})$. Besides, the concavity changes at $x=0, x=-\sqrt{3}, x=\sqrt{3}$ and thus f has inflection points at $(-\sqrt{3},f(-\sqrt{3}))$, $(0,f(0)),(\sqrt{3},f(\sqrt{3}))$.

Hence, the inflection points are $(-\sqrt{3},-\sqrt{3})$, $(0,0),(\sqrt{3},\sqrt{3})$.

e) $f'(x)=\frac{2}{3}x^3-4x=\frac{2}{3}x(x^2-6)$, $f''(x)=2x^2-4=2(x^2-2)$.

Hence, the function is concave up on $(-\infty,-\sqrt{2})\cup(\sqrt{2},\infty)$, concave down on $(-\sqrt{2},\sqrt{2})$ and it has inflection points at $x=\pm\sqrt{2}$.

Theorem (Necessary condition for Inflection point):

If the graph of f has an inflection point at $(c,f(c))$, then either $f''(c)=0$ or $f''(c)$ does not exist. The converse of the theorem need not be true.

For instance, if $f(x)=x^4+1, f''(x)=12x^2$ such that $f''(0)=0$ but $(0,f(0))=(0,1)$ is not its inflection point because $f''(x)=12x^2\geq 0, \forall x\in R$ which means there is no change in sign of $f''(x)$ at $x=0$.

Examples (Applications of the condition):

1. If the point $(3,6)$ is an inflection point of $f(x)=x^4+ax^3+b$, find the values of the constants a and b.

Solution: Since $f(x)=x^4+ax^3+b$ is a polynomial and $(3,6)$ is an inflection point on the graph of f, by the above necessary condition, $f''(3)=0$.

That is $f''(x)=12x^2+6ax \Rightarrow f''(3)=0 \Rightarrow 18a+108=0 \Rightarrow a=-6$. Besides, $(3,6)$ is an inflection point means $f(3)=6 \Rightarrow 81-6(27)+b=6 \Rightarrow b=87$.

2. If $f(x)=ax^5+bx^3+2$ has a relative extreme value at $x=\sqrt{2}$ and an inflection point at $(1,-5)$, find the values of the constants a and b.

Solution: Since every polynomial has relative extreme values only at its critical number, $x=\sqrt{2}$ a critical number for $f(x)=ax^5+bx^3+2$.

So, $f'(\sqrt{2})=0 \Rightarrow 5a(\sqrt{2})^4+3b(\sqrt{2})^2=0 \Rightarrow 20a+6b=0 \Rightarrow 10a+3b=0$.

Besides, the point $(1,-5)$ is its inflection point.

Thus, $f(1)=-5 \Rightarrow a+b+2=-5 \Rightarrow a+b=-7$.

Collect and solve the two equations as follow:

$$\begin{cases} 10a+3b=0 \\ a+b=-7 \end{cases} \Rightarrow 10a+3(-7-a)=0 \Rightarrow 7a-21=0 \Rightarrow 7a=21 \Rightarrow a=3$$

Again, $a+b=-7 \Rightarrow 3+b=-7 \Rightarrow b=-10$.

Summarizing the main concepts: .The following functions are carefully selected to generalize the most important concepts.

For each of the following functions, find

i) All the critical numbers

ii) The interval where f(x) is increasing and decreasing

iii) The relative maximum and minimum values of f

iv) The interval where f(x) is concave up and concave down

v) The inflection point(s)

$a^*)\ f(x)=8+5x^4-x^5$ $\quad b^*)\ f(x)=\frac{x^2+2x+4}{x}$ $\quad c)\ f(x)=\frac{x^4}{4}-x^3$

$d)\ f(x)=\frac{1-x}{e^x}$ $\quad e)\ f(x)=\frac{2x^2-1}{x^4}$ $\quad f)\ f(x)=x^4-6x^2+9$

$g)\ f(x)=\frac{x^4}{4}-\frac{9x^2}{2}+\frac{1}{4}$ $\quad h)\ f(x)=\frac{x^3}{3}-2x^2-5x$

Solution: Apply the respective derivative tests.

a) Here, $f'(x)=0 \Rightarrow 20x^3-5x^4=0 \Rightarrow 5x^3(4-x)=0 \Rightarrow x=0, x=4$

Construct sign chart for $f'(x)$ using the critical numbers as poles.

Intervals	$x<0$	$0\le x\le 4$	$x>4$
$5x^3$	$---$	$+++$	$+++$
$4-x$	$+++$	$+++$	$---$
$f'(x)=5x^3(4-x)$	$---$	$+++$	$---$

Relative Maximum and Relative Minimum: By FDT;

Relative minimum: Rel.min.val $=f(0)=8$.

Relative maximum: Rel.max.val $=f(4)=264$.

Intervals of Monotone: By FDT:

Interval of increasing: $0\le x\le 4$ or $[0,4]$

Interval of Decreasing: $(-\infty,0]\cup[4,\infty)$

The intervals of concave up and concave down

$f''(x)=0 \Rightarrow 60x^2-20x^3=0 \Rightarrow 20x^2(3-x)=0 \Rightarrow x=0, x=3$

Construct sign chart for $f''(x)$ using $x=0, x=3$ as poles.

Intervals	$x<0$	$0<x<3$	$x>3$
$20x^2$	+++	+++	+++
$3-x$	+++	+++	---
$f''(x)=20x^2(3-x)$	+++	+++	---

By SDT: Interval of concave up: $\{x: x<3\}$ or $\{x: x\in(-\infty,3)\}$

Interval of concave down: $\{x: x>3\}$ or $\{x: x\in(3,\infty)\}$

Inflection point: $(3, f(3))=(3,170)$

$$b^*)\ f'(x)=\frac{x^2-4}{x^2}=\frac{(x-2)(x+2)}{x^2}=0\Rightarrow(x-2)(x+2)=0\Rightarrow x=2, x=-2$$

Hence, the critical numbers are $x=2, x=-2$ and the function is undefined at $x=0$. Now construct the sign chart using these numbers.

Intervals	$x\le -2$	$-2\le x<0$	$0<x\le 2$	$x\ge 2$
$x-2$	---	---	---	+++
$x+2$	---	+++	+++	+++
x^2	+++	+++	+++	+++
$f'(x)=\frac{(x-2)(x+2)}{x^2}$	+++	---	---	+++

From the sign chart, $f'(x)\ge 0$ on $-\infty<x\le -2$ and $2\le x<\infty$ and $f'(x)\le 0$ on $-2\le x<0$ and $0<x\le 2$. Therefore, f is increasing on $(-\infty,-2]\cup[2,\infty)$ and decreasing on $[-2,0)\cup(0,2]$. Besides, the sign of $f'(x)$ changes from positive to negative at $x=-2$. This means f has relative maximum at $x=-2$, that is $f(-2)=-2$ is relative maximum value. Again the sign of $f'(x)$ changes from negative to positive at $x=2$. This means f has relative minimum at $x=2$, that is $f(2)=6$ is relative minimum value. To find the interval of concavity and inflection point, lets use SDT.

That is $f''(x)=\frac{8}{x^3}$. Then f is concave up on the interval where $f''(x)>0$. That is $f''(x)=\frac{8}{x^3}>0 \Rightarrow x^3>0 \Rightarrow x>0$.

Thus, the function is concave up on the interval $(0,\infty)$.

Again, it is concave down on $f''(x)<0$. That is $f''(x)=\frac{8}{x^3}<0 \Rightarrow x<0$

Thus, the function is concave down on the interval $(-\infty,0)$.

Since the point $x=0$ is not in the domain, there is no inflection point.

c) $f'(x)=x^3-3x^2=0 \Rightarrow x^2(x-3)=0 \Rightarrow x=0, x=3$. Hence, the critical numbers are $x=0, x=3$. Now construct the sign chart.

Interval	$-\infty<x<0$	$0<x<3$	$3<x<\infty$
x^2	+++	+++	+++
$x-3$	---	---	+++
$f'(x)=x^2(x-3)$	---	---	+++

Therefore, the function is decreasing on $(-\infty,3]$ and increasing on $[3,\infty)$. Besides, the sign of $f'(x)$ changes from negative to positive at $x=3$. This means f has relative minimum at $x=3$, that is $f(3)=-\frac{27}{4}$ is relative minimum value. Since $f'(x)$ does not change at $x=0$, f has no relative minimum or relative maximum value at $x=0$. To find the interval of concavity and inflection point, lets use SDT.

$f''(x)=3x^2-6x=0 \Rightarrow 3x(x-2)=0 \Rightarrow x=0, x=2$. Then f is concave up on the interval where $f''(x)>0$ and concave down on the interval where $f''(x)<0$. Since there are two numbers where $f''(x)=0$, it is better to construct the sign chart for $f''(x)$.

Interval	$-\infty<x<0$	$0<x<2$	$2<x<\infty$
$3x$	$---$	$+++$	$+++$
$x-2$	$---$	$---$	$+++$
$f''(x)=3x(x-2)$	$+++$	$---$	$+++$

Therefore, the function is concave up on $(-\infty,0)\cup(2,\infty)$ and concave down on $(0, 2)$. Since the concavity changes at $x=0$ and $x=2$, f has inflection points at $x=0$ and $x=2$. Hence, the inflection points are $(0,f(0))=(0,0)$ and $(2,f(2))=(2,-4)$.

d) Here, $f(x)=\frac{1-x}{e^x} \Rightarrow f'(x)=\frac{-e^x-(1-x)e^x}{(e^x)^2}=\frac{(x-2)e^x}{(e^x)^2}=\frac{x-2}{e^x}$

Critical number: $f'(x)=0 \Rightarrow \frac{x-2}{e^x}=0 \Rightarrow x-2=0 \Rightarrow x=2$.

Construct sign chart for $f'(x)$ using the critical numbers as poles.

Intervals	$x<2$	$x>2$
$x-2$	$---$	$+++$
e^x	$+++$	$+++$
$f'(x)=\frac{x-2}{e^x}$	$---$	$+++$

Relative Maximum and Relative Minimum: By FDT;

Relative minimum: Rel.min $=f(2)=\frac{1-2}{e^2}=-\frac{1}{e^2}$.

Relative maximum: There is no relative maximum value.

Intervals of Monotone: By FDT for monotone;

Interval of increasing: $[2,\infty)$

Interval of Decreasing: $(-\infty,2]$

Intervals of concavity and Inflection point:

First: Find the points where $f''(x)=0$ or $f''(x)$ does not exist.

That is $f''(x)=0 \Rightarrow \frac{e^x-(x-2)e^x}{e^{2x}}=0 \Rightarrow \frac{3-x}{e^x} \Rightarrow 3-x=0 \Rightarrow x=3$

Construct sign chart for $f''(x)$ using $x=3$ as a pole.

Intervals	$x<3$	$x>3$
$3-x$	+++	---
e^x	+++	+++
$f''(x)=\frac{3-x}{e^x}$	+++	---

By SDT: Interval of concave up: $\{x:x<3\}$ or $\{x:x\in(-\infty,3)\}$

Interval of concave down: $\{x:x>3\}$ or $\{x:x\in(3,\infty)\}$

Inflection point: IP at $(3,f(3))=(3,-\frac{2}{e^3})$ since the concavity changes at $x=3$.

e) $f'(x)=4x^3-12x=0 \Rightarrow 4x(x-\sqrt{3})(x+\sqrt{3})=0 \Rightarrow x=0, x=\pm\sqrt{3}$

5.10.4 Solving Optimization Problems

Optimization is the process of optimizing (that is either minimizing or maximizing) the values of a given applied problem. Usually, in an ordinary situation of optimization problems, there are two types of functions. The **objective (Primary)** and **constraint (Secondary)** functions. The objective (Primary) function is the function that we want to maximize or minimize its value where as the constraint (Secondary) function is the one that relates two or more of the variables in the problem using a given constant.

Generally, optimization problems are of the form

$$\begin{cases} f(x,y): \textbf{Objective function} \\ g(x,y) = Constant - \textbf{Constraint function} \end{cases}$$

Guides line for Solving Optimization Problems: Many students find optimization problems intimidating because they are "word" problems, and there does not appear to be a pattern to these problems and even these problems demands rigorous use of previous knowledge. However, if you are patient you can minimize your anxiety and maximize your success with these problems by practicing more problems. Let me tell you one fact. No one is perfect to do all types of problems in Mathematics but through practice and many years of experience everybody becomes a good problem solver in his or her field of specialization. To be effective in solving applied maximum and minimum problems, one has to keep the following guide lines in mind:

Step-1: Understanding the problem: Identify the given and the required quantities. If appropriate, draw a sketch or diagram of the problem to be solved.

Step-2: Labeling: Select variables to be used and carefully label your picture or diagram with these variables. This step is very important because it leads to the creation of mathematical equations.

Step-3: Equation formation: Write down all equations which are related to your problem from the sketch. Here, to write equations you are advised to recall known geometric formula from geometry.

Step-4: Identification: Identify the constraint equation and the objective function. That means select the function which you are asked to maximize or minimize. Usually, the objective function involves two or more variables.

So, using the constraint equation, change the objective function into a function of one variable only.

Step-5: Computation of Critical numbers: Find the critical numbers of the objective function in the *feasible domain* for the variables.

Decision: Using the first or second derivative test for extreme, decide whether the value at each critical numbers is a maximum or minimum.

Here, under a number of problems which range in difficulty from average to challenging are selected and solved.

Examples:

1. Find two positive numbers whose sum is 16 and whose product is as large as possible.

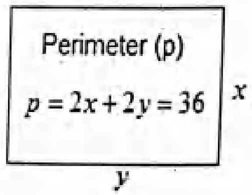
Solution: Here, let x and y be the two numbers such that $x+y=16$ (constraint) and their product is $p(x,y)=xy$ (objective function). But $x+y=16 \Rightarrow y=16-x$. So, the product becomes only a function of one variable. That is $p(x)=x(16-x)=16x-x^2$. Now lets find the critical numbers. $p'(x)=16-2x=0 \Rightarrow x=8$. A gain $y=16-x \Rightarrow y=8$.

Besides, $p''(x)=-2 \Rightarrow p''(8)<0$. Thus, by Second Derivative Test, the product is maximum when $x=8, y=8$.

2. Find the dimensions of a rectangle with maximum area whose perimeter is 36.

Solution: Let the dimensions of the rectangle be x and y. Then, the area is given by $A(x,y)=xy$. But from the given perimeter, x and y are related as

$p=2x+2y=36 \Rightarrow x+y=18 \Rightarrow y=18-x$.



Now writing the area as a function of x only gives us

$A(x)=x(18-x)=18x-x^2$. Here, to maximize the area, first find the critical number of this area function. That is $A'(x)=18-2x=0 \Rightarrow x=9$. But by the second derivative test, $A''(x)=-2 \Rightarrow A''(9)=-2$ and thus we have maximum value at this critical number. Besides, $x=9 \Rightarrow y=9$.

Therefore, the maximum area of the rectangle is obtained when the rectangle is a square of sides $x = 9$ and the maximum area is $A = 9x9 = 81$.

3. Suppose you are asked to design an open rectangular box with a square base and having a volume of $32cm^3$. Find the dimensions of the box that will minimize the amount of materials needed in the construction.

Solution: Let x be the base and y its height. Then, the volume of the box is given by the formula: $V = x^2y = 32$. Besides, its surface area is $S = x^2 + 4xy$

Now, solve for y from the volume and substitute in the surface area.

That is $V = x^2y = 32 \Rightarrow y = \frac{32}{x^2}$. Put $y = \frac{32}{x^2}$ in the surface area.

That is $S = x^2 + 4xy = x^2 + 4x(\frac{32}{x^2}) \Rightarrow S = x^2 + \frac{128}{x}$.

Critical number: Using derivative, find the critical numbers of S.

$S' = 0 \Rightarrow 2x - \frac{128}{x^2} = 0 \Rightarrow \frac{2x^3 - 128}{x^2} = 0 \Rightarrow 2x^3 - 128 = 0 \Rightarrow x^3 = 64 \Rightarrow x = 4$

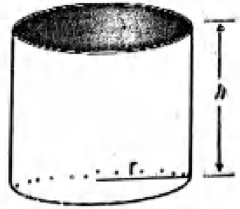
Besides, we have $y = \frac{32}{x^2} \Rightarrow y = \frac{32}{16} = 2$

Test the critical number: Using SDT: $S'' = 2 + \frac{256}{x^3} \Rightarrow S''(4) > 0$.

Hence $x = 4$ and $y = 2$ give minimum surface area.

4. A closed cylindrical can with a volume 128π cubic units is to be formed. Find the dimensions of the cylinder that will minimize the amount of material to use.

Solution: The amount of material to be used means just the material needed to cover the total surface area which is the sum of the two base areas (top and bottom bases) and the lateral surface area of the cylinder. Hence, the total surface area is given by $A = 2\pi r^2 + 2\pi rh$. Besides, we are given the volume is $V = \pi r^2 h = 128\pi$.



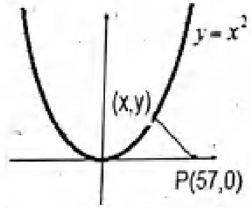
Here, $\pi r^2 h = 128\pi \Rightarrow h = \dfrac{128}{r^2}$ and thus $A = 2\pi r^2 + \dfrac{256\pi}{r}$. So,

$$A'(r) = 4\pi r - \frac{256\pi}{r^2} = 0 \Rightarrow r^3 - 64 = 0 \Rightarrow r = 4 \text{ and } h = \frac{128}{r^2} \Rightarrow h = 8.$$

(Verify that the second derivative is positive at the critical number)

5. Find a point on the curve $y = x^2$ closest to the point $P(57,0)$?

Solution: Let (x, y) be any point on the given curve as shown in the figure below.



Then, the distance between the given point and the required point (x, y) is given by $d = \sqrt{(x-57)^2 + (y-0)^2}$. Since the point (x, y) is on the curve, we have $y = x^2$. Hence, the distance formula becomes $d = \sqrt{(x-57)^2 + x^4}$. Now to minimize the distance, let's find the critical number.

$d' = \dfrac{2x^3 + x - 57}{\sqrt{(x-57)^2 + x^4}} = 0 \Rightarrow 2x^3 + x - 57 = 0 \Rightarrow x = 3$. Besides, from $y = x^2$, we have $y = 9$. Hence, the point on the curve with minimum distance from $P(57,0)$ than any other points is $(3,9)$. (Establish using first or second derivative test).

6. In an apartment building that has 100 rooms, all are rented out when the monthly rent is $r = 900$ dollars. Suppose that once room becomes vacant with each 10 dollars increase in rent and each occupied room costs 80 dollars in maintenance per month. What rent r maximizes the monthly profit?

Solution: Let n be the number of 10 dollars increase in rent. Then, the monthly profit is given by $P(n) = (100-n)(900+10n-80) = 82000 + 180n - 10n^2$.

Hence, the critical number becomes $P'(n) = 180 - 20n = 0 \Rightarrow n = 9$.

So, the monthly profit is maximized if the rent is $r = 900 + 10(9) = 990$ dollar.

7. Two particles A and B are in motion in the xy plane. Their coordinates at each instance of time is given by $x_A = 4t, y_A = t^2, x_B = 2t, y_B = 6$. Find the minimum possible distance between the particles.

Solution: At any time the square of the distance between them is given by

$S = D^2 = (x_A - x_B)^2 + (y_A - y_B)^2 = 4t^2 + (t^2 - 6)^2 = t^4 - 8t^2 + 36$.

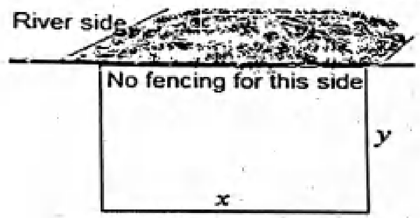
So, $S'(t) = 4t^3 - 16t = 0 \Rightarrow 4t(t-2)(t+2) = 0 \Rightarrow t = 0, t = 2, t = -2$.

Since time is non-negative, only $t = 0$ and $t = 2$ are valid.

Hence, the minimum distance is $D = \sqrt{20} = 2\sqrt{5}$km (if distance is in km).

6. A rectangular plot of land is bounded by a fence on three sides and by a straight river on the fourth side. Find the dimensions of the field with maximum area that can be enclosed with $1000 ft$ of fencing.

Solution: Let x and y be the dimensions of the plot as shown below.



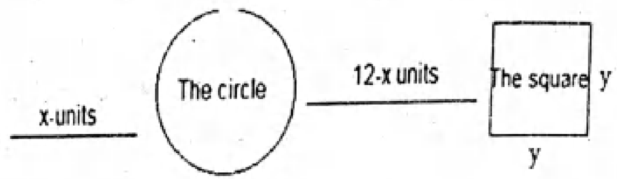
The length of the fencing material (which is the perimeter of the plot mathematically) is given by $x + 2y = 1000$ and the area becomes $A = xy$.

But from $x + 2y = 1000$, we have $y = 500 - \frac{x}{2}$. So, $A(x) = 500x - \frac{x^2}{2}$. Now find the critical number. $A'(x) = 500 - x = 0 \Rightarrow x = 500$ and thus $y = 500 - \frac{x}{2} = 250$.

Therefore the dimension of the rectangle is $500 ft$ by $250 ft$. Besides, $A''(x) = -1 < 0$ which means (by SDT), we have maximum area and the maximum area is $125{,}000 ft^2$.

7. A wire of length 12m can be bent into a circle, bent into a square or cut in to two pieces to make both a circle and a square. How much wire should be used for the circle if the total area enclosed by the figure (s) is to be maximum?

Solution: Let the length of the wire used for the circle be x .Then, the length used for the square will be $12-x$.



Observe that the circumference of the circle is x and the perimeter of the square is $12-x$. $x=2\pi r \Rightarrow r=\dfrac{x}{2\pi}, 4y=12-x \Rightarrow y=\dfrac{12-x}{4}$.

So, $A_T=\pi r^2+y^2=\dfrac{\pi x^2}{4\pi^2}+\left(\dfrac{12-x}{4}\right)^2=\dfrac{x^2}{4\pi}+\left(\dfrac{12-x}{4}\right)^2$, $x\in[0,12]$

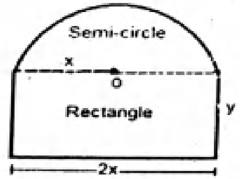
Hence, $\dfrac{dA_T}{dx}=\dfrac{x}{2\pi}-\dfrac{12-x}{8}=0 \Rightarrow x=\dfrac{12\pi}{4+\pi}\in[0,12]$

$A_T(0)=9,\ A_T(\dfrac{12\pi}{4+\pi})=\dfrac{36\pi+9}{(4+\pi)^2},\ A_T(12)=\dfrac{36}{\pi}$

The maximum area occurs when $x=12$ and the figure formed is only a circle.

8. A window in the shape of a rectangle with a semicircle attached at the top is constructed with a perimeter of 12m. Find the dimensions that allow maximum amount of light to enter.

Solution: Let the radius of the semi-circle be x such that one side of the rectangle will be the diameter which is $2x$. Let the other side be y (refer the diagram below).



Then the perimeter of the window becomes the sum of the three sides of the rectangle and the semicircle which is $P=2x+\pi x+2y=12$ and the area of the figure becomes the sum of the area of the rectangle and the area of the semi-circle which is given by $A = 2xy + \frac{\pi x^2}{2}$.

Here, $2x + \pi x + 2y = 12 \Rightarrow 2y = 12 - 2x - \pi x$. Hence, the area becomes

$$A(x) = \frac{\pi x^2}{2} + x(12 - 2x - \pi x).$$

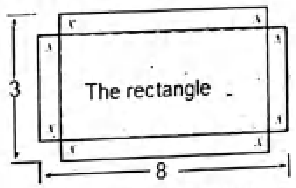
Observe that the maximum amount of light will enter when the total area of the window is maximum. Now find the critical number. That is

$A'(x) = 12 - (\pi + 4)x = 0 \Rightarrow x = \frac{12}{\pi + 4}$ and from

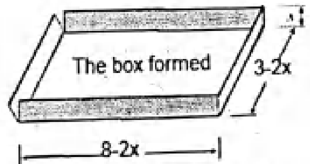
$2y = 12 - 2x - \pi x$, we get $y = \frac{12}{\pi + 4}$. Besides, $A''(x) = -(\pi + 4) < 0$. Therefore, we have maximum value at the critical numbers.

9. An open rectangular box is to be formed from a rectangular piece of card board of dimension 3inches by 8 inches by cutting out squares of equal size from the four corners and bending up the sides. What size of squares should be cut to obtain a box with maximum volume?

Solution: Let x be the side of the square to be cut out. Then the dimensions of the box formed will be $8 - 2x$ by $3 - 2x$ as shown in the diagrams below. Hence, the volume becomes $V = x(8 - 2x)(3 - 2x)$.



Before cutting the squares


After cutting the squares

As a measure of lengths $8 - 2x$ and $3 - 2x$ has to be non-negative, that is $8 - 2x \geq 0 \Rightarrow x \leq 4$ and $3 - 2x \geq 0 \Rightarrow x \leq 3/2$. This means the domain for the problem is $0 \leq x \leq 3/2$. Now find the critical numbers.

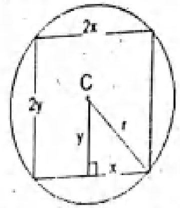
$V' = 4(3x - 2)(x - 3) = 0 \Rightarrow x = 2/3, x = 3$. Since $x = 3$ is out of the domain, the only critical number is $x = 2/3$. Besides, $V(0) = 0, V(3/2) = 0, V(2/3) = 200/27$

Hence, a box with maximum volume is obtained when $x = 2/3$.

9. Find the dimensions and maximum area of a rectangle that can be inscribed in a circle of radius $r = 10$ units.

Solution: Assume the center C of the circle is at the origin (to simplify computation).

Let the sides of the rectangle be $2x$ and $2y$ as shown in the figure below.



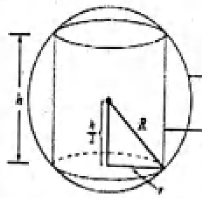
By Pythagoras Theorem, $x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 100$. Besides, the area is $A = 4xy$. But from $x^2 + y^2 = 100$ we have $y = \sqrt{100 - x^2}$. Hence, the area becomes $A(x) = 4x\sqrt{100 - x^2}$. Now find the critical numbers.

$$A'(x) = 4\sqrt{100 - x^2} - \frac{4x^2}{\sqrt{100 - x^2}} = 0 \Rightarrow x = 5\sqrt{2} \text{ and } y = \sqrt{100 - x^2} \Rightarrow y = 5\sqrt{2}.$$

Hence, for the rectangle to have maximum area its dimension is $10\sqrt{2}$ by $10\sqrt{2}$

13. Find the maximum volume and dimensions (radius and height) of a right-circular cylinder that could be situated symmetrically inside a sphere of radius $R = 1$.

Solution: Let r and h denotes the radius and height of the right-circular cylinder as shown in the diagram below.

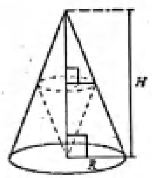


Then, volume of the cylinder is given by $V = \pi r^2 h$ but using Pythagoras Theorem, $r^2 + \left(\frac{h}{2}\right)^2 = R^2 = 1 \Rightarrow r^2 = 1 - \frac{h^2}{4}$. So,

$V = \pi r^2 h = \pi(1 - \frac{h^2}{4})h = \pi(h - \frac{h^3}{4})$ for $0 \le h \le 2$.

Now differentiate with respect to h and find the critical number of this volume function. That is $V' = \pi(1 - \frac{3h^2}{4}) = 0 \Rightarrow 1 - \frac{3h^2}{4} = 0 \Rightarrow h^2 = \frac{4}{3} \Rightarrow h = \frac{2}{\sqrt{3}}$. But $V(0) = 0, V(2) = 0, V(\frac{2}{\sqrt{3}}) = \frac{4\pi}{3\sqrt{3}}$. Therefore, the maximum volume is $V = \frac{4\pi}{3\sqrt{3}}$ which is obtained when $h = \frac{2}{\sqrt{3}}$ and $r = \sqrt{\frac{2}{3}}$

14. Find the maximum volume and dimensions (radius and height) of a right-circular cone that could be placed (inscribed) upside-down in a right-circular cone of radius $R = 6$ and height $H = 8$ as shown in the diagram below.



Solution: Let r and h denotes the radius and height of the inscribed right-circular cone. Then, by using similarity of the smaller (the top) triangle and the larger triangle, we have $\frac{8-h}{r} = \frac{8}{6} \Rightarrow h = 8(1 - \frac{r}{6})$.

Besides, from geometry the volume of a cone of height h and radius r is given by $V = \frac{1}{3}\pi r^2 h = \frac{8}{3}\pi(r^2 - \frac{r^3}{6})$, for $0 \le r \le 6$. Now differentiate with respect to r and find the critical number of this volume function. That is

$V' = \frac{8}{3}\pi(2r - \frac{r^2}{2}) = 0 \Rightarrow r = 0, r = 4$.

But $V(0) = 0, V(6) = 0, V(4) = \frac{128\pi}{9}$. Therefore, the maximum volume is $\frac{128\pi}{9}$ which is obtained when the radius and height of the cone are $r = 4$ and h=8/3.

5.10.5 Indeterminate Forms and L' Hôpital's Rules

While solving limit problems, it is common to encounter expressions of the form $\frac{0}{0}, \frac{\infty}{\infty}, \infty-\infty, 0.\infty, 0^0, \infty^0, 1^\infty$. Expressions of such form are called *Indeterminate forms* because we cannot decide their value by simple inspection unless some analysis is used. The technique of finding the values of such indeterminate forms is known as *L'Hopital's Rule*. In what follows, we are going to state L'Hopital's rule for indeterminate of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and then generalize the results for other indeterminate forms.

Case-I: Indeterminate of the form $\frac{0}{0}$:

Suppose $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=0$ and assume that $g'(x)\neq 0$ in an open interval containing $x=a$. Then, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Note that L'Hopital's rule can also be applied to evaluate one sided limits like $\lim_{x\to a^+}\frac{f(x)}{g(x)}$, $\lim_{x\to a^-}\frac{f(x)}{g(x)}$, or $\lim_{x\to \pm\infty}\frac{f(x)}{g(x)}$.

Examples: Evaluate the following limits

$a)\ \lim_{x\to 0}\frac{\sin 3x}{2x}$ $\quad b)\ \lim_{x\to 2}\frac{\ln(5-x^2)}{x-2}$ $\quad c)\ \lim_{x\to \frac{\pi}{2}}\frac{\sin 2x}{4x^2-\pi^2}$ $\quad d)\ \lim_{x\to 0}\frac{e^{5x}-1}{\tan x}$

$e)\ \lim_{x\to 1}\frac{\tan \pi x}{\ln x}$ $\quad f)\ \lim_{x\to 0}\frac{1-\cos^2 x}{x^2}$ $\quad g)\ \lim_{x\to 1}\frac{\ln x}{x-\sqrt{x}}$ $\quad h)\ \lim_{x\to 1}\frac{x^3-1}{2\sqrt{x}-2x}$

Solution: Each of the limits are the form $0/0$. So, let's apply L'Hopital's rule.

$$a)\ \lim_{x\to 0}\frac{\sin 3x}{2x}=\lim_{x\to 0}\frac{(\sin 3x)'}{(2x)'}=\lim_{x\to 0}\frac{3\cos 3x}{2}=\frac{3}{2}$$

$$b)\ \lim_{x\to 2}\frac{\ln(5-x^2)}{x-2}=\lim_{x\to 2}\frac{(\ln(5-x^2))'}{(x-2)'}=\lim_{x\to 2}\frac{-2x}{5-x^2}=-4$$

c) $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin 2x}{4x^2-\pi^2} = \lim\limits_{x\to\frac{\pi}{2}} \dfrac{2\cos 2x}{8x} = \dfrac{-1}{2\pi}$

d) $\lim\limits_{x\to 0} \dfrac{e^{5x}-1}{\tan x} = \lim\limits_{x\to 0} \dfrac{(e^{5x}-1)'}{(\tan x)'} = \lim\limits_{x\to 0} \dfrac{5e^{5x}}{\sec^2 x} = 5$

e) $\lim\limits_{x\to 1} \dfrac{\tan \pi x}{\ln x} = \lim\limits_{x\to 1} \dfrac{(\tan \pi x)'}{(\ln x)'} = \lim\limits_{x\to 1} \dfrac{\pi \sec^2 \pi x}{\frac{1}{x}} = \pi$

f) $\lim\limits_{x\to 0} \dfrac{1-\cos^2 x}{x^2} = \lim\limits_{x\to 0} \dfrac{(1-\cos^2 x)'}{(x^2)'} = \lim\limits_{x\to 0} \dfrac{2\sin x\cos x}{2x} = \lim\limits_{x\to 0} \dfrac{\sin 2x}{2x} = 1$

g) $\lim\limits_{x\to 1} \dfrac{\ln x}{x-\sqrt{x}} = \lim\limits_{x\to 1} \dfrac{2\sqrt{x}}{x(2\sqrt{x}-1)} = 2$

h) $\lim\limits_{x\to 1} \dfrac{x^3-1}{2\sqrt{x}-2x} = \lim\limits_{x\to 1} \dfrac{3x^2}{\frac{1}{\sqrt{x}}-2} = \lim\limits_{x\to 1} \dfrac{3x^2\sqrt{x}}{1-2\sqrt{x}} = -3$

i) $\lim\limits_{x\to 1} \dfrac{4^x-3^x-1}{x-1} = \lim\limits_{x\to 1} \dfrac{4^x \ln 4 - 3^x \ln 3}{1} = 4\ln 4 - 3\ln 3$

Remark: Sometimes application of L'Hopital's rule to indeterminate forms may lead again to indeterminate forms. In such cases, it is possible to apply the rule repeatedly as far as the condition of the rule is satisfied.

Examples: Evaluate the following limits

a) $\lim\limits_{x\to 0} \dfrac{3x-\tan 3x}{x^3}$ b) $\lim\limits_{x\to -2} \dfrac{(x+2)^3}{\sin(x+2)-x-2}$ c) $\lim\limits_{x\to 1} \dfrac{3e^{x^3-1}-3}{x^3-1}$

d) $\lim\limits_{x\to 0} \dfrac{e^{3x}-2\cos x+e^{-3x}}{x\sin x}$ e) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2}$ f) $\lim\limits_{x\to 0} \dfrac{x^2}{1-\cos^2 x}$

g) $\lim\limits_{x\to 0} \dfrac{e^x-e^{-x}-2x}{x-\sin x}$ h) $\lim\limits_{x\to 1} \dfrac{\ln x - x + 1}{x^3-3x+2}$ i) $\lim\limits_{x\to 0} \dfrac{\ln(e^{3x}-3x)}{x^2}$

Solution:

a) Here, the limit is an indeterminate of the form $\frac{0}{0}$. So, applying L'Hopital's rule gives $\lim_{x\to 0}\frac{3x-\tan 3x}{x^3}=\lim_{x\to 0}\frac{3-3\sec^2 3x}{3x^2}=\lim_{x\to 0}\frac{1-\sec^2 3x}{x^2}$. But,

$\lim_{x\to 0}\frac{1-\sec^2 3x}{x^2}$ is again an indeterminate of the form $\frac{0}{0}$. Thus, let's apply the rule for the second time. That is

$$\lim_{x\to 0}\frac{1-\sec^2 3x}{x^2}=\lim_{x\to 0}\frac{-6\sec^2 3x\tan 3x}{2x}=-3(\lim_{x\to 0}\sec^2 3x)(\lim_{x\to 0}\frac{\tan 3x}{x})=-9.$$

Therefore, by twice application of L'Hopital's rule, we get

$$\lim_{x\to 0}\frac{3x-\tan 3x}{x^3}=\lim_{x\to 0}\frac{3-3\sec^2 3x}{3x^2}=\lim_{x\to 0}\frac{-6\sec^2 3x\tan 3x}{2x}=-9$$

b) Here, applying L'Hopital's rule three times gives

$$\lim_{x\to -2}\frac{(x+2)^3}{\sin(x+2)-x-2}=\lim_{x\to -2}\frac{3(x+2)^2}{\cos(x+2)-1}\qquad (\frac{0}{0}\ form)$$

$$=\lim_{x\to -2}\frac{6(x+2)}{-\sin(x+2)}\qquad (\frac{0}{0}\ form)$$

$$=\lim_{x\to -2}\frac{6}{-\cos(x+2)}=-6$$

c) $\lim_{x\to 1}\frac{3e^{x^3-1}-3}{x^3-1}=\lim_{x\to 1}\frac{(3e^{x^3-1})(3x^2)}{3x^2}=\lim_{x\to 1}3e^{x^3-1}=3$

d) Here, the limit is an indeterminate of the form $\frac{0}{0}$. Then,

$$\lim_{x\to 0}\frac{e^{3x}-2\cos x+e^{-3x}}{x\sin x}=\lim_{x\to 0}\frac{3e^{3x}+2\sin x-3e^{-3x}}{\sin x+x\cos x}=\lim_{x\to 0}\frac{9e^{3x}+2\cos x+9e^{-3x}}{2\cos x-x\sin x}=10$$

e) $\lim_{x\to 0}\frac{1-\cos x}{3x^2}=\lim_{x\to 0}\frac{\sin x}{6x}=\lim_{x\to 0}\frac{\cos x}{6}=\frac{1}{6}$

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$f)\ \lim_{x\to 0}\frac{x^2}{1-\cos^2 x}=\lim_{x\to 0}\frac{2x}{2\sin x\cos x}=\lim_{x\to 0}\frac{x}{\sin 2x}=\lim_{x\to 0}\frac{1}{2\cos 2x}=\frac{1}{2}$

$g)\ \lim_{x\to 0}\frac{e^x-e^{-x}-2x}{x-\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}-2}{1-\cos x}=\lim_{x\to 0}\frac{e^x-e^{-x}}{\sin x}=\lim_{x\to 0}\frac{e^x+e^{-x}}{\cos x}=2$

$h)\ \lim_{x\to 1}\frac{\ln x-x+1}{x^3-3x+2}=\lim_{x\to 1}\frac{1/x-1}{3x^2-3}=\lim_{x\to 1}\frac{-1/x^2}{6x}=-\frac{1}{6}$

$i)\ \lim_{x\to 0}\frac{\ln(e^{3x}-3x)}{x^2}=\lim_{x\to 0}\frac{3e^{3x}-3}{2x(e^{3x}-3x)}=\lim_{x\to 0}\frac{9e^{3x}}{2(e^{3x}-3x)+2x(3e^{3x}-3)}=\frac{9}{2}$

Case-II: Indeterminate of the form $\frac{\infty}{\infty}$:

Suppose $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=\infty$ and assume that $g'(x)\neq 0$ in an open interval containing $x=a$. Then, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$.

Examples: Evaluate the following limits

$a)\ \lim_{x\to+\infty}\frac{\ln(x^2+1)}{\ln x}$ $b)\ \lim_{x\to+\infty}\frac{e^{2x}}{x^3}$ $c)\ \lim_{x\to\infty}\frac{\ln(e^{3x}+x^2)}{2x+3}$ $d)\ \lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\cos x)}{\ln(\tan x)}$

Solution: Each of the limits are the form ∞/∞. So, let's apply L'Hopital's rule.

$a)\ \lim_{x\to+\infty}\frac{\ln(x^2+1)}{\ln x}=\lim_{x\to+\infty}\frac{2x^2}{x^2+1}=\lim_{x\to+\infty}\frac{4x}{2x}=\lim_{x\to+\infty}\frac{4}{2}=2$

$b)\ \lim_{x\to+\infty}\frac{e^{2x}}{x^3}=\lim_{x\to+\infty}\frac{2e^{2x}}{3x^2}=\lim_{x\to+\infty}\frac{4e^{2x}}{6x}=\lim_{x\to+\infty}\frac{8e^{2x}}{6}=\frac{8}{6}\lim_{x\to+\infty}e^{2x}=\infty$

$c)\ \lim_{x\to\infty}\frac{\ln(e^{3x}+x^2)}{2x+3}=\lim_{x\to\infty}\frac{3e^{3x}+2x}{2(e^{3x}+x^2)}=\lim_{x\to\infty}\frac{9e^{3x}+2}{6e^{3x}+4x}$

$=\lim_{x\to\infty}\frac{27e^{3x}}{18e^{3x}+4}=\lim_{x\to\infty}\frac{81}{54}=\frac{3}{2}$

$d)\ \lim_{x\to\frac{\pi}{2}^-}\frac{\ln(\cos x)}{\ln(\tan x)}=\lim_{x\to\frac{\pi}{2}^-}\frac{-\tan x}{\cot x\sec^2 x}=\lim_{x\to\frac{\pi}{2}^-}(-\sin^2 x)=-1$

Case III: Other Indeterminate forms

Indeterminate of the form $0.\infty$ **:** Suppose $\lim_{x\to a} f(x)=0$, $\lim_{x\to a} g(x)=\infty$. Then, the limit of the product $f(x)g(x)$ will be of the form $0.\infty$. This means that there is a struggle between the values of f and g as x approaches a. If f wins the struggle, the limit will be 0, if g wins the struggle, the limit will be ∞ and if they compromise each other, the limit will be some finite number between 0 and ∞. In either cases, it is not possible to determine the value and thus it is an indeterminate form. Then, the indeterminate form of this type is determined by changing it in to indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

This is possible by using the conversion relation $f(x)g(x)=\dfrac{f(x)}{\dfrac{1}{g(x)}}$.

Besides, from the property of limit, we have $\lim_{x\to a} f(x)=0 \Leftrightarrow \lim_{x\to a}\dfrac{1}{f(x)}=\infty$.

Therefore, $\lim_{x\to a} f(x)=0,\ \lim_{x\to a} g(x)=\infty \Rightarrow \lim_{x\to a} f(x)g(x)=\lim_{x\to a}\dfrac{f(x)}{\dfrac{1}{g(x)}}\to\dfrac{0}{0}$ (*form*)

Or $\lim_{x\to a}\dfrac{1}{f(x)}=\infty,\ \lim_{x\to a} g(x)=\infty \Rightarrow \lim_{x\to a} f(x)g(x)=\lim_{x\to a}\dfrac{g(x)}{\dfrac{1}{f(x)}}\to\dfrac{\infty}{\infty}$ (*form*).

After converting to these forms, to the basic indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we can apply L'Hopital's rules that we discussed in case I and case II above.

Examples:

1. Evaluate the following limits.

a) $\lim_{x\to 0} x^2\ln x^2$ *b)* $\lim_{x\to\infty} x\tan\left(\dfrac{3}{x}\right)$ *c)* $\lim_{x\to 0^+}\sin x\ln(\sin x)$ *d)* $\lim_{x\to 0^+}\sin x\ln x$

Solution:

a) Here, $\lim_{x\to 0} x^2=0$, $\lim_{x\to 0}\ln x^2=-\infty$. This means the problem is indeterminate of the form $0.\infty$.

Thus, using the above conversion $\lim\limits_{x\to 0} x^2 \ln x^2 = \lim\limits_{x\to 0} \dfrac{\ln x^2}{\dfrac{1}{x^2}} \to (\dfrac{\infty}{\infty} form)$.

Then, $\lim\limits_{x\to 0} x^2 \ln x^2 = \lim\limits_{x\to 0} \dfrac{\ln x^2}{\dfrac{1}{x^2}} = \lim\limits_{x\to 0} \dfrac{(\ln x^2)'}{(\dfrac{1}{x^2})'} = \lim\limits_{x\to 0} (-x^2) = 0$

b) Here, $\lim\limits_{x\to\infty} x = \infty$, $\lim\limits_{x\to\infty} \tan(\frac{3}{x}) = 0$ .It is an indeterminate of the form $0.\infty$.

Thus, $\lim\limits_{x\to\infty} x\tan(\frac{3}{x}) = \lim\limits_{x\to\infty} \dfrac{\tan(\frac{3}{x})}{1/x} \to (0/0 form)$.

Then, $\lim\limits_{x\to\infty} x\tan(\frac{3}{x}) = \lim\limits_{x\to\infty} \dfrac{\tan(\frac{3}{x})}{1/x} = \lim\limits_{x\to\infty} \dfrac{\sec^2(\frac{3}{x})(-3/x^2)}{(-1/x^2)} = \lim\limits_{x\to\infty} 3\sec^2(\frac{3}{x}) = 3$

c) $\lim\limits_{x\to 0^+} \sin x \ln(\sin x) = \lim\limits_{x\to 0^+} \dfrac{\ln(\sin x)}{1/\sin x} = \lim\limits_{x\to 0^+} \dfrac{\cos x}{\sin x}(-\dfrac{\sin^2 x}{\cos x}) = \lim\limits_{x\to 0^+} (-\sin x) = 0$

d) Here, $\lim\limits_{x\to 0^+} \sin x = 0$, $\lim\limits_{x\to 0^+} \ln x = \infty$.So, the problem is of the form $0.\infty$.

Hence, $\lim\limits_{x\to 0^+} \sin x \ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{\sin x}} = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\csc x}$.......$(\dfrac{\infty}{\infty}$ form).

Therefore by applying L' Hopital's rule twice gves

$$\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\csc x} = \lim_{x\to 0^+} \frac{-\sin x \tan x}{x}$$
$$= \lim_{x\to 0^+} -(\cos x \tan x + \sin x \sec^2 x) = 0.$$

2. For what value of k, does $\lim\limits_{x\to 5} \dfrac{3x^2 + kx - 15}{x-5}$ exist?

Solution: Since $x = 5$ makes the denominator zero, for the limit to exist we have to get a factor $x-5$ in the numerator $3x^2 + kx - 15$ so that to cancel the denominator term and evaluate the limit.

In short, $x = 5$ has to be the zero of $3x^2 + kx - 15$.

That is $3x^2 + kx - 15 \Rightarrow 75 + 5k - 15 = 0 \Rightarrow 5k = -60 \Rightarrow k = -12$.

Indeterminate of the form $\infty-\infty$:

Suppose $\lim_{x\to a} f(x)=\lim_{x\to a} g(x)=\infty$. Then, $\lim_{x\to a}[f(x)-g(x)]$ results an indeterminate of the form $\infty-\infty$. An indeterminate of this form can be reduced to an indeterminate of the form $0/0$ or ∞/∞ using the conversion relation

$$f(x)-g(x)=\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$$

Examples: Evaluate the following limits.

a) $\lim_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right)$ b) $\lim_{x\to\frac{\pi^-}{4}}(\sec 2x-\tan 2x)$ c) $\lim_{x\to 2^+}\left(\frac{1}{\ln(x^2-3)}-\frac{1}{4x-8}\right)$

d) $\lim_{x\to 0}\left(\frac{1}{x^2}-\csc^2 x\right)$ e) $\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)$ f) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

Solution:

a) Here, the problem is indeterminate of the form $\infty-\infty$.

Hence, using the above conversion, we have $xe^{\frac{1}{x}}-x=x(e^{\frac{1}{x}}-1)=\frac{e^{\frac{1}{x}}-1}{1/x}$.

Therefore, $\lim_{x\to\infty}\left(xe^{\frac{1}{x}}-x\right)=\lim_{x\to+\infty}\frac{e^{\frac{1}{x}}-1}{1/x}=\lim_{x\to+\infty}\frac{e^{\frac{1}{x}}(-1/x^2)}{-1/x^2}=1$

b) Here, $\lim_{x\to\frac{\pi^-}{4}}\sec 2x=\infty$, $\lim_{x\to\frac{\pi^-}{4}}\tan 2x=\infty$. So, it is of the form $\infty-\infty$.

Hence, using the above conversion, we have

$$\sec 2x-\tan 2x=\frac{\frac{1}{\tan 2x}-\frac{1}{\sec 2x}}{\frac{1}{\tan 2x\sec 2x}}=\frac{1-\sin 2x}{\cos 2x}.$$

Therefore, $\lim_{x\to\pi^-/4}(\sec 2x-\tan 2x)=\lim_{x\to\pi^-/4}\frac{1-\sin 2x}{\cos 2x}=\lim_{x\to\pi^-/4}\frac{-2\cos 2x}{-2\sin 2x}=0$

c) Similarly, $\lim_{x\to 2^+}\frac{1}{\ln(x^2-3)}=\infty$, $\lim_{x\to 2^+}\frac{1}{4x-8}=\infty$. So, it is of the form $\infty-\infty$.

$$\lim_{x\to 2^+}\left(\frac{1}{\ln(x^2-3)}-\frac{1}{4x-8}\right)=\lim_{x\to 2^+}\frac{4x-8-\ln(x^2-3)}{(4x-8)\ln(x^2-3)} \qquad (\tfrac{0}{0}\ form)$$

$$=\lim_{x\to 2^+}\frac{4-\frac{2x}{x^2-3}}{4\ln(x^2-3)+(4x-8)\frac{2x}{x^2-3}} \qquad (\tfrac{0}{0}\ form)$$

$$=\lim_{x\to 2^+}\frac{\frac{-2(x^2-3)+4x^2}{(x^2-3)^2}}{\frac{16x}{x^2-3}+(4x-8)\frac{-6-2x^2}{(x^2-3)^2}}=\frac{7}{16}$$

d) Here, $\lim_{x\to 0}\frac{1}{x^2}=\infty$, $\lim_{x\to 0}\csc^2 x=\infty$ .So, the problem is indeterminate of the form $\infty-\infty$. Hence, using the above conversion relation we have

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\csc^2 x\right)=\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\left(\frac{\sin^2 x-x^2}{x^2\sin^2 x}\right) \quad (\tfrac{0}{0}\ form)$$

$$=\lim_{x\to 0}\frac{\sin 2x-2x}{2x\sin^2 x+x^2\sin 2x} \qquad (\tfrac{0}{0}\ form)$$

$$=\lim_{x\to 0}\frac{2\cos 2x-2}{2\sin^2 x+4x\sin 2x+2x^2\cos 2x} \qquad (\tfrac{0}{0}\ form)$$

$$=\lim_{x\to 0}\frac{-4\sin 2x}{6\sin 2x+12x\cos 2x-4x^2\sin 2x} \qquad (\tfrac{0}{0}\ form)$$

$$=\lim_{x\to 0}\frac{-8\cos 2x}{24\cos 2x-32x\sin 2x-8x^2\cos 2x}=-\frac{1}{3}$$

e) This is indeterminate of the form $\infty-\infty$. Hence, using the above conversion

$$\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{1}{x-1}\right)=\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{x-1}{x\ln x+x-1}=\lim_{x\to 1}\frac{1}{\ln x+2}=\frac{1}{2}$$

Case-IV: Indeterminate of the form $0^0, \infty^0, 1^\infty$:

These types of indeterminate forms could be obtained from the evaluation of limits of the form $h(x) = f(x)^{g(x)}$. To determine such limits, first change these forms into either forms that we discussed above. This is done as follow:

$$h(x) = f(x)^{g(x)} \Rightarrow \ln h(x) = g(x)\ln f(x)$$
$$\Rightarrow \lim \ln h(x) = \lim g(x)\ln f(x)$$
$$\Rightarrow \ln(\lim h(x)) = \lim g(x)\ln f(x) = k \ (Say)$$
$$\Rightarrow \lim h(x) = e^k \Rightarrow \lim h(x) = \lim f(x)^{g(x)} = e^k$$

Examples:

1. Evaluate the following limits

a) $\lim\limits_{x\to 2^-}(5-x^2)^{\frac{1}{2-x}}$ b) $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$ c) $\lim\limits_{x\to\infty}(e^{6x}-5)^{\frac{1}{x}}$ d) $\lim\limits_{x\to 0^+} x^x$

e) $\lim\limits_{x\to 0}(1-3x)^{\frac{1}{x}}$ f) $\lim\limits_{x\to 1^+} x^{\frac{1}{1-x}}$ g) $\lim\limits_{x\to 0^+}(\sin x)^{\ln x}$ h) $\lim\limits_{x\to 0^+}(2x+1)^{\cot x}$

Solution:

a) This is an indeterminate of the form 1^∞. So,

$$h(x) = (5-x^2)^{\frac{1}{2-x}} \Rightarrow \ln h(x) = \frac{\ln(5-x^2)}{2-x}$$

$$\Rightarrow \lim_{x\to 2^-}\ln h(x) = \lim_{x\to 2^-}\frac{\ln(5-x^2)}{2-x} \qquad (\tfrac{0}{0}\ form)$$

$$\Rightarrow \lim_{x\to 2^-}\ln h(x) = \lim_{x\to 2^-}\frac{2x}{5-x^2} = 4 \Rightarrow \ln\lim_{x\to 2^-} h(x) = 4 \Rightarrow \lim_{x\to 2^-} h(x) = e^4$$

$$\Rightarrow \lim_{x\to 2^-} h(x) = \lim_{x\to 2^-}(5-x^2)^{\frac{1}{2-x}} = e^4$$

b) Let $y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{\ln x}{x} \Rightarrow \lim\limits_{x\to\infty}\ln y = \lim\limits_{x\to\infty}\frac{\ln x}{x}$ $\frac{\infty}{\infty}$ form.

Thus, by L'Hopital's rule, $\lim\limits_{x\to\infty}\ln y = \ln\lim\limits_{x\to\infty} y = 0 \Rightarrow \lim\limits_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1$

c) $\lim_{x\to\infty}(e^{6x}-5)=\infty, \lim_{x\to\infty} 1/x=0$. It is an indeterminate of the form ∞^0.

$$\lim_{x\to\infty}\ln h(x)=\lim_{x\to\infty}\frac{\ln(e^{6x}-5)}{x} \quad (\frac{\infty}{\infty}\ form)$$

$$\Rightarrow \lim_{x\to\infty}\ln h(x)=\lim_{x\to\infty}\frac{6e^{6x}}{e^{6x}-5} \quad (\frac{\infty}{\infty}\ form)$$

$$\Rightarrow \lim_{x\to\infty}\ln h(x)=\lim_{x\to\infty}\frac{36e^{6x}}{6e^{6x}}=6$$

$$\Rightarrow \ln\lim_{x\to\infty}h(x)=6 \Rightarrow \lim_{x\to\infty}h(x)=e^6 \Rightarrow \lim_{x\to\infty}\left(e^{6x}-5\right)^{\frac{1}{x}}=e^6$$

d) Let $y=x^x \Rightarrow \ln y = x\ln x=\frac{\ln x}{\frac{1}{x}} \Rightarrow \lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}}\ \frac{\infty}{\infty}$ form.

Thus, $\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln x}{1/x}=\lim_{x\to0^+}\frac{1/x}{-1/x^2}=0 \Rightarrow \lim_{x\to0^+}x^x=e^0=1$

e) Let $y=(1-3x)^{\frac{1}{x}} \Rightarrow \ln y=\frac{\ln(1-3x)}{x} \Rightarrow \lim_{x\to0}\ln y=\lim_{x\to0}\frac{\ln(1-3x)}{x}\ \frac{0}{0}$ form

By L'Hopital's rule, $\lim_{x\to0}\frac{-3}{1-3x}=-3 \Rightarrow \lim_{x\to0}(1-3x)^{\frac{1}{x}}=e^{-3}$.

f) Let $y=x^{\frac{1}{1-x}} \Rightarrow \ln y=\frac{\ln x}{1-x} \Rightarrow \lim_{x\to1^+}\ln y=\lim_{x\to1^+}\frac{\ln x}{1-x}\ \frac{0}{0}$ form

By L'Hopital's rule, $\lim_{x\to1^+}\ln y=\lim_{x\to1^+}-1/x=-1$.

Thus, $\lim_{x\to1^+}\ln y=\ln\lim_{x\to1^+}y=-1 \Rightarrow \lim_{x\to1^+}x^{\frac{1}{1-x}}=e^{-1}=\frac{1}{e}$

g) Let $y=(\sin x)^{\ln x} \Rightarrow \ln y=\frac{\ln(\sin x)}{\ln x} \Rightarrow \lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln(\sin x)}{\ln x}$ ($\frac{\infty}{\infty}$ form)

By L'Hopital's rule, $\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{x\cos x}{\sin x}=\lim_{x\to0^+}\frac{\cos x-x\sin x}{\cos x}=1$.

Thus, $\lim_{x\to0^+}\ln y=\ln\lim_{x\to0^+}y=1 \Rightarrow \lim_{x\to0^+}(\sin x)^{\ln x}=e$.

2. Using L'Hopital's rule, evaluate the following limits

a) $\lim_{x\to 0^+}\left(5^x\right)^{\frac{1}{\sin x}}$ b) $\lim_{x\to\infty}\left(e^{3x}+5x\right)^{\frac{1}{x}}$ c) $\lim_{x\to 0}\left(e^{3x}-3x\right)^{\frac{1}{x^2}}$ d) $\lim_{x\to\infty}\left(3^x+4^x\right)^{\frac{1}{x}}$

Solution:

a) Let $y=\left(5^x\right)^{\frac{1}{\sin x}} \Rightarrow \ln y=\dfrac{\ln 5^x}{\sin x} \Rightarrow \lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\dfrac{\ln 5^x}{\sin x}$ $\left(\dfrac{0}{0}\text{ form}\right)$.

Thus, $\lim_{x\to 0^+}\ln y=\ln\lim_{x\to 0^+}y=\ln 5 \Rightarrow \lim_{x\to 0^+}\left(5^x\right)^{\frac{1}{\sin x}}=e^{\ln 5}=5$

b) $\ln y=\dfrac{\ln(e^{3x}+5x)}{x} \Rightarrow \lim_{x\to+\infty}\ln y=3 \Rightarrow \lim_{x\to+\infty}\left(e^{3x}+5x\right)^{\frac{1}{x}}=e^3.$

c) Let $y=\left(e^{3x}-3x\right)^{\frac{1}{x^2}} \Rightarrow \ln y=\dfrac{\ln(e^{3x}-3x)}{x^2} \Rightarrow \lim_{x\to 0}\ln y=\lim_{x\to 0}\dfrac{\ln(e^{3x}-3x)}{x^2}$ $\left(\dfrac{0}{0}\right)$

By L'Hopital's rule, $\lim_{x\to 0}\ln y=\lim_{x\to 0}\dfrac{3e^{3x}-3}{2x(e^{3x}-3x)}=\dfrac{9}{2} \Rightarrow \lim_{x\to 0}\left(e^{3x}-5x\right)^{\frac{1}{x^2}}=e^{9/2}.$

d) Let $y=\left(3^x+4^x\right)^{\frac{1}{x}} \Rightarrow \ln y=\dfrac{\ln(3^x+4^x)}{x} \Rightarrow \lim_{x\to+\infty}\ln y=\lim_{x\to+\infty}\dfrac{\ln(3^x+4^x)}{x}$ $\left(\dfrac{\infty}{\infty}\right)$

By L'Hopital's rule, $\lim_{x\to+\infty}\ln y=\lim_{x\to+\infty}\dfrac{3^x\ln 3+4^x\ln 4}{3^x+4^x}=\lim_{x\to+\infty}\dfrac{(3/4)^x\ln 3+\ln 4}{(3/4)^x+1}=\ln 4$

Thus, $\lim_{x\to+\infty}\ln y=\ln\lim_{x\to+\infty}y=\ln 4 \Rightarrow \lim_{x\to+\infty}\left(3^x+4^x\right)^{\frac{1}{x}}=e^{\ln 4}=4.$

3. Find the values of the constants a and b such that $\lim_{x\to 0}\dfrac{a+\cos bx}{x^2}=-8$.

Solution: Since $\lim_{x\to 0}x^2=0$, for the limit to exist the limit of the numerator must also be zero. That is $\lim_{x\to 0}(a+\cos bx)=0 \Rightarrow a+1=0 \Rightarrow a=-1$.

Hence, by L'Hopital's rule twice, we have

$$\lim_{x\to 0}\frac{a+\cos bx}{x^2}=\lim_{x\to 0}\frac{-1+\cos bx}{x^2}=\lim_{x\to 0}\frac{-b\sin bx}{2x}=\lim_{x\to 0}\frac{-b^2\cos bx}{2}=\frac{-b^2}{2}.$$

But using the given value we have

$$\lim_{x\to 0}\frac{a+\cos bx}{x^2}=\frac{-b^2}{2}=-8 \Rightarrow b^2=16 \Rightarrow b=\pm 4.$$

Review Problems on Chapter-5

1. If $g(0)=3$, $g'(0)=-2$ where $f(x)=e^{4x}g(x)$, then $f'(0)=$ _______
2. Find the equations of the tangent and normal lines at the given point.
a) $f(x)=e^{1-x}$ at $x=1$ b) $f(x)=x^2e^x-2xe^x+2e^x$ at $x=1$
c) $f(x)=x^{\sin x}$ at $x=\pi/2$ d) $f(x)=3e^{2x}-x\ln(x+1)$ at $x=0$
Answer: a) $y=2-x$ b) $y=ex$ c) $y=x$ d) $y=6x+3$
3. Find the formula for the nth derivatives of $f(x)=e^{-2x}+\ln(1-3x)$.
4. Find the equations of the tangent and normal lines at the given point.
a) $f(x)=3e^{-x}\sin(2x)-4x$; at $x=0$ b) $f(x)=x\cos(\sqrt{2}x)$ at $(0,0)$
c) $y^4+3xy+x^4=5$; at $(1,1)$ d) $\sin(xy)=y$; at $(\frac{\pi}{2},1)$
5. Find the equation of the tangent line to $f(x)=(e^{5x}-\sin 2x)^4$ at $x=0$.
6. If $f(x)=x^2+ax+b$ and $g(x)=cx-x^2$ have a common tangent line at the point $(1,0)$, find the constants a,b and c. **Answer** : $a=-3, b=2, c=1$
7*. Let $h(x)=(f\circ g)(x)$ and $g(x)=\sqrt{25-x^2}$ where $f(4)=5, f'(4)=8$. Give the equation of the tangent line to the graph of h at $x=3$. **Answer:** $6x+y=23$
8. Find the value of a if the tangent line to the graph of $f(x)=x^3-ax^2+3x+1$ at $(1,2)$ is parallel to x-axis. **Answer:** $a=3$
9. Find the points where the curve $25x^2+16y^2+200x-160y+400=0$ has vertical or horizontal tangent line.
Answer : Horizontal at (-4,3),(-4,10) and vertical at (0,5),(-8,5)
10. Find the points on the curve $f(x)=x^2+\frac{1}{x}$ where tangent line to the graph of f has a slope of $m=-3$. **Answer:** $(-1,0)$ & $(\frac{1}{2},\frac{9}{4})$
11. Gas is escaping from a spherical balloon at a rate of $2cm^3/s$. How fast is the surface area changing when the radius is 12cm? **Answer** : $-1/3$

12. A spherical balloon is losing its air at a rate of $2cm^3/\sec$. How fast is the radius of balloon shrinking when the volume of the balloon is $36\pi cm^3$?

13. Calculate $(f^{-1})'(c)$ of the following functions.

$a)\ f(x)=x^3+6x+2; c=2$

$b)\ f(x)=x^5+4x^3+4x+2; c=2$

$c)\ f(x)=x^3+2x-1; c=2$

$d)\ f(x)=6\ln x;\ \ c=0$

$e)\ f(x)=x\ln x\ \ c=3e^3$

$f)\ f(x)=x+\sin x;\ c=0$

$g)\ f(x)=3x-\frac{1}{x^3}, x<0, c=-2$

$h)\ f(x)=\frac{x+6}{x-2}, x>2; c=3$

$i)\ f(x)=\frac{1}{x+1}, 0<x\le 1; c=\frac{1}{4}$

$j)\ f(x)=x^3-\frac{4}{x}, x>0; c=6$

$k)\ f(x)=\tan x, -\frac{\pi}{4}\le x\le\frac{\pi}{4};\ c=\frac{\sqrt{3}}{3}$

$l)\ f(x)=\int_0^x\sqrt{1+t^4}; c=f(1)$

Answer : $a)\ \frac{1}{6}$ $\quad b)\ \frac{1}{4}$ $\quad c)\ \frac{1}{5}$ $\quad d)\ \frac{1}{6}$ $\quad e)\ \frac{1}{4}$ $\quad f)\ 1$

$g)\ \frac{1}{6}$ $\quad h)\ -2$ $\quad i)\ -16$ $\quad j)\ \frac{1}{13}$ $\quad k)\ \frac{3}{4}$ $\quad l)\ \frac{1}{\sqrt{2}}$

14. Find the value of the constant k so that $f(x)=\frac{x}{x^2+k}$ has a relative extreme value at $x=3$. **Answer** : $k=9$

15. The rate of a certain chemical reaction at any time t is modeled by the equation $R(t)=k(4-t), k>0$. On which time interval is the rate of reaction maximum?

16*. Let $f(x)=\ln\left(\frac{\pi}{2}+\tan^{-1}x\right)$. Find $(f^{-1})'(\ln\frac{\pi}{3})$. **Answer :** $\frac{4\pi}{9}$

17. For $f(x)=x^5+4x^2+8x-7$ with $f(1)=6, (f^{-1})'(6)$ is_____. **Answer:** $\frac{1}{21}$

18. For the following functions, find all the critical numbers in the whole domain and give the absolute maximum and minimum values on the given interval.

$a)\ f(x)=x^2+3x^{\frac{2}{3}}-6\ \ \text{on}[-8,8]$

$b)\ f(x)=\begin{cases}6x^2+12x+1, -1\le x\le 0\\ x^2-4x+1, 0<x\le 3\end{cases}\quad \text{on}[-1,3]$

Answer : $a)$ Max $=70$ & Min $=-6$ $\ b)$ Max $=1$ & Min $=-5$

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19. Let t $f(x) = \ln(5-\sqrt{3x+7})$. Then, find $(f^{-1})'(0)$.

20*. Determine the values of k such that the $h(x) = kx + \sin x$ has an inverse.

Answer : $k \le -1$ or $k \ge 1$

21*. The following functions are carefully selected to generalize main concepts of the chapter. For each of them, please try to find:

a) All the critical numbers
b) The intervals of increasing and decreasing
c) The relative maximum and relative minimum values
d) The intervals where the graph of f is concave up and concave down
e) The inflection points (if any)

$a) f(x) = 2x^3 - 9x^2 + 4$ $\quad b) f(x) = x^4 - 4x^3 + 4x^2$ $\quad c) f(x) = x^2(x-2)$

$d) f(x) = \frac{x^4}{4} - \frac{9x^2}{2}$ $\quad e) f(x) = x + \frac{4}{x}$ $\quad f) f(x) = \frac{x}{x^2-1}$

$g) f(x) = x^4 + 4x^3 + 24$ $\quad h)(x) = x^3 - 3x^2 - 9x$ $\quad i) f(x) = (x^2-3)^2$

$j) f(x) = 2x^4 - 4x^3 + 3$

22. Simplify the following expressions:

$a)\ \cos(\sin^{-1} 2x)$ $\quad b)\ \sec(\tan^{-1} x)$ $\quad c)\ \cos(\sin^{-1} 2x)$ $\quad d)\ \sec(\tan^{-1} x)$

$e)\ \cos(\cot^{-1} x^2)$ $\quad f)\ \tan(\cos^{-1} \frac{x}{2})$ $\quad g)\ \cos(\cot^{-1} x^2)$ $\quad h)\ \tan(\cos^{-1} \frac{x}{2})$

Answer : $a)\ \sqrt{1-4x^2}$ $\quad b)\ \sqrt{x^2+1}$ $\quad c)\ \sqrt{1-4x^2}$ $\quad d)\ \sqrt{x^2+1}$

$e)\ \frac{x^2}{\sqrt{x^4+1}}$ $\quad f)\ \frac{\sqrt{4-x^2}}{x}$ $\quad g)\ \frac{x^2}{\sqrt{x^4+1}}$ $\quad h)\ \frac{\sqrt{4-x^2}}{x}$

23. Find the numerical values of the following expressions

$a)\ \sinh(\ln 3)$ $\quad b)\ \coth(\ln 5)$ $\quad c)\ \operatorname{sech}(\ln\sqrt{2})$ $\quad d)\ \cosh^{-1}(\sqrt{2})$

Answer : $a)\ \frac{4}{3}$ $\quad b)\ \frac{13}{12}$ $\quad c)\ \frac{2\sqrt{2}}{3}$ $\quad d)\ \ln(1+\sqrt{2})$

24. A land owner wishes to use 3miles of fencing material to enclose an isosceles triangular region. What should be the lengths of the sides so that to enclose maximum area?

Answer : 1 *mile each side*

25. Find two nonnegative numbers whose sum is 9 and so that the product of one number and the square of the other is maximum. **Answer** : 3 & 6

26. Find the largest volume of a right circular cone that can be inscribed in a sphere of radius 3. **Answer** : $h = 32\pi/3$

27. What is the maximum area of a rectangle whose perimeter is 60m? **Answer** : $A = 225m^2$

28*. Suppose $f(x) = x^5 + x^3 + x + 2$ and $(g \circ f)(x) = x$. If $F(x) = f(2g(x))$, then for $c = g(5)$, $(F^{-1})'(c) =$ _____ **Answer:** $\frac{62}{3}$

29. Given $f(\sqrt{x^3 - 2}) = \dfrac{x^4}{12} + \dfrac{5x^3}{3}$. Then, evaluate $f'(5)$. **Answer:** 20

30. If $f'(x) = 3x^2 f(x)$ and $f(0) = \frac{1}{e}$, then $f''(1) =$ _____ **Answer:** 12

31. Using L'Hopital's rule, evaluate the following limits

a) $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ b) $\lim\limits_{x\to 0} \dfrac{x^2 + 3\sin x}{x}$ c) $\lim\limits_{x\to 0} \dfrac{\cos 2x - \cos x}{\sin^2 x}$

d) $\lim\limits_{x\to \infty} x \ln\left(\dfrac{x+3}{x-3}\right)$ e) $\lim\limits_{x\to 0}\left(\dfrac{1}{x^2} - \cot^2 x\right)$ f) $\lim\limits_{x\to 1} \csc(\pi x)\ln x$

g) $\lim\limits_{x\to 1} \dfrac{2e^{x^2-1} - 2}{x^2 - 1}$ h) $\lim\limits_{x\to 1} \dfrac{\ln x^2}{x - 1}$ i) $\lim\limits_{x\to 0^+} \dfrac{\tan x}{x^2}$ j) $\lim\limits_{x\to 0^+} \dfrac{\sin x}{\sqrt{x}}$

k) $\lim\limits_{x\to 1} \dfrac{\ln x}{\tan \pi x}$ l) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^2 \tan x}$ m) $\lim\limits_{x\to 0} \dfrac{x\tan x}{1 - \cos x}$ n) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\cos x}{2x - \pi}$

o) $\lim\limits_{x\to 1} \dfrac{x^x - x}{x - 1 - \ln x}$ p) $\lim\limits_{x\to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x\ln(1-x)}$ q) $\lim\limits_{x\to 0} \dfrac{e^x + 2\sin x - e^{-x} - 4x}{x^5}$

r) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{2^{\sin x} - 2}{\ln(\sin x)}$ s) $\lim\limits_{x\to \infty}\left(\cos\dfrac{1}{x^2}\right)^{x^4}$ t) $\lim\limits_{x\to 0} \dfrac{(1+x)^{\frac{1}{x}} - e}{x}$ u) $\lim\limits_{x\to 0}\left(\dfrac{1}{\sin^2 x} - \dfrac{1}{x^2}\right)$

Answer: a) 2 b) 3 c) $-3/2$ d) 6 e) $2/3$ f) $-1/\pi$ g) 3
h) 2 i) $+\infty$ j) 0 k) $1/\pi$ l) $1/5$ m) 2 n) $-1/2$
o) 2 p) $-2/3$ q) $1/30$ r) $2\ln 2$ s) $e^{-1/2}$ t) $-e/2$ u) $1/3$

32. The values of a and b such that $f(x) = x^3 + ax^2 + b$ will have relative extreme value at (2,3) are ____&______respectively. **Answer:** -3 & 7

33. Let $f(\theta) = 1 - 3\cos\theta + \frac{6}{\sin\theta}$. Give the minimum value of f on $(0, \frac{\pi}{2})$.

35. If $\lim_{x\to 0}\left(e^{ax}-5x\right)^{\frac{1}{x}}=e^{-2}$, $\lim_{x\to\infty}\left(e^{bx}-5x\right)^{\frac{1}{x}}=e^{6}$, find the values of the constants a and b.

Answer : $a=3,=6$

36. Find the values of the constants a and b such that $\lim_{x\to 0}\frac{a+\cos bx}{x^2}=-4$.

Answer : a = -1, b = $\pm 2\sqrt{2}$

37. An open rectangular box with square base is to be made from 48 ft.^{2} of material. What dimensions will result in a box with the largest possible volume ?

Answer : $s=4$, $h=2$, $V\max=32$

38. If $g'(4)=48$, then evaluate $\lim_{x\to 4}\frac{g(x)-g(4)}{x^3-5x^2+4x}$. **Answer:** 16

39. A closed cylindrical can is to hold $40\pi m^3$ liquid. The material for the top and bottom costs $20 birr/m^2$ and the material for the side costs $8 birr/m^2$. Find the radius and height of the can to construct the most economical can.

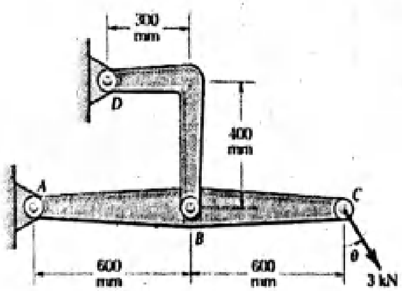
Answer : $r=2m$, $h=10m$

40. An engineer designed the frame below to support a load of $L=3$kN which may be applied at an angle of θ where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. If the pins at A and B are designed to support a maximum force, then

a) What is the maximum magnitude of the force at B? For what value of angle θ it will be maximum? In this case, what is the force at A?

b) What is the maximum magnitude of the force at A? For what value of angle θ it will be maximum? In this case, what is the force at B?

Answer: a) $F_{max}=7.5$kN, $F_A=5.4$kN b) $F_{max}=6$kN, $F_B=6.7$kN



CHAPTER-6

INTEGRATION AND ITS APPLICATIONS

6.1 The Concept of Indefinite Integrals

Definition: Let f be continuous on an interval I. Then, the function F is said to be *anti-derivative* of f on I if and only if $F'(x) = f(x), \forall x \in I$. For instance, the functions $F_1(x) = \frac{x^3}{3}$, $F_2(x) = \frac{x^3}{3} + 1$, $F_3(x) = \frac{x^3}{3} - 3$ are all anti-derivatives of $f(x) = x^2$ because $F'_1(x) = F_2'(x) = F_3'(x) = x^2 = f(x)$.

Definition: The set of all anti-derivatives of a function f is called *indefinite integrals of f*. The process of finding indefinite integrals or anti-derivatives of a function is called *integration* (sometimes known as *anti-differentiation*).

Notation: $\int f(x)dx = F(x) + c$.

Here, $f(x)$ is called *integrand* and the symbol $\int$ is known as an *integral sign*. The constant c is called *constant* of integration.

Theorem:

1. If $F(x)$ is anti-derivative of $f(x)$, then $F(x)+c$ is also anti-derivative of $f(x)$ for any constant c. This Theorem tells us that anti-derivative of a function is not *unique* unless and otherwise some pre-condition is given to determine the constant of integration. This is because for instance if we take $F(x) = x^2$, $G(x) = x^2 + 3$, $H(x) = x^2 - 7$ are anti-derivatives of $f(x) = 2x$. So, as there are infinitely many constants, there are infinite anti-derivatives.

2. If $F(x)$ and $G(x)$ are anti-derivative of $f(x)$ on I, then $F(x) = G(x) + c, \forall x \in I$. This means that any two anti-derivatives of a function differ by a constant.

Basic Integration Rules: In order to be effective in finding the integrals of different functions, it is useful to grasp rules that facilitate the integration process. Let f and g be any two integrable functions (functions whose integral exists) and k be any constant. Then,

$i)\int(f+g)(x)dx=\int f(x)dx+\int g(x)dx$ $\quad ii)\int(f-g)(x)dx=\int f(x)dx-\int g(x)dx$

$iii)\int f'(x)dx=f(x)$ $\quad iv)\int kf(x)dx=k\int f(x)dx$ and $\int kdx=kx+c$

$v)\int x^n dx=\frac{x^{n+1}}{n+1}+c, n\neq -1$ and $\int \frac{1}{x}dx=\ln|x|+c$

6.2 Techniques of Integration

6.2.1 Integration by Substitution

From chain rule, for two differentiable functions f and g, we have seen that $[f\circ g]'(x)=f'(g(x))g'(x)$.

Now we want to find $\int[f\circ g]'(x)dx=\int f'(g(x))g'(x)dx$ as a reverse process of chain rule. In such cases, we usually substitute $u=g(x)$, $\frac{du}{dx}=g'(x)$ so that the integral becomes $\int f'(g(x))g'(x)dx=\int f'(u)du=f(u)=f(g(x))$.

Here, since the method depends on substitution to evaluate the integral, the method is called integration by substitution (simply substitution method). Substitution techniques involve the introduction of a function that changes the form of the integrand into a simpler integrand than the original. To use this method effectively, first identify the function $u=g(x)$ and find $du=g'(x)dx$.

Then, express the integral in terms of u and integrate with respect to u.

Finally, replace all $u's$ in the final result by $g(x)$.

If your choice of $u=g(x)$ leads to simpler integral, you are on the 'red carpet, Oh! nice to continue' but if your choice leads to a more complicated integral, you are on the 'red light, Shit! not nice to continue', change your choice.

It is not an easy task to identify which one to be substituted but it is a skill that can be developed through practice and experience.

Basic hints in Substitution Method:

Form-1:

Integrals of the form $\int h(x)f(g(x))dx$ **OR** $\int \frac{h(x)}{f(g(x))}dx$ **OR** $\int \frac{f(g(x))}{h(x)}dx$**:**

If $h(x)$ RESEMBLES $g'(x)$, that means $h(x) = kg'(x)$ or $h(x)$ is a scalar multiple of the derivative of $g(x)$, use the substitution $u = g(x)$.

Examples:

Evaluate the following integrals using substitution.

a) $\int 32x(x^2+4)^7 dx$ b) $\int 4x^3\cos(x^4+1)dx$ c) $\int \sin x e^{\cos x} dx$

d) $\int (x+1)\sqrt{x^2+2x}dx$ e) $\int (2x^3+x)(x^4+x^2)^{49}dx$ f) $\int e^x(e^x+1)^7 dx$

g) $\int \frac{x^3}{(4x^4+1)^2}dx$ h) $\int \frac{\sin 2x}{(5+\cos 2x)^3}dx$ i) $\int \frac{3x^2+4}{\sqrt{x^3+4x+1}}dx$

j) $\int (2-\frac{1}{x^2})(2x+\frac{1}{x})dx$ k) $\int (1+\frac{1}{x})\sin(x+\ln x)dx$

Solution:

a) Look carefully, if we take $h(x) = 32x$, $f(x) = x^7$, $g(x) = x^2+4$, the integrand $I(x) = 32x(x^2+4)^7$ is of the form $h(x)f(g(x)) = 32x(x^2+4)^7$. Besides, $h(x) = 32x$ resembles $g'(x) = 2x$. That is $h(x) = 16.g'(x)$.

So, by the above hint, use the substitution, $u = g(x) = x^2+4 \Rightarrow du = 2xdx$.

Therefore, $\int 32x(x^2+4)^7 dx = \int 16u^7 du = 2u^8 + c = 2(x^2+4)^8 + c$.

b) Look ! if we take $h(x) = 4x^3$, $f(x) = \cos x$, $g(x) = x^4+1$, the integrand $I(x) = 4x^3\cos(x^4+1)$ is of the form $h(x)f(g(x)) = 4x^3\cos(x^4+1)$. Besides, $h(x) = 4x^3$ resembles $g'(x) = 4x^3$. That is $h(x) = g'(x)$.

So, by the above hint, use the substitution, $u = g(x) = x^4+1 \Rightarrow du = 4x^3dx$.

Therefore, $\int 4x^3\cos(x^4+1)dx = \int \cos u du = \sin u + c = \sin(x^4+1) + c$.

c) Here, $u=\cos x, du=-\sin x dx$.

Then, $\int \sin x e^{\cos x} dx = -\int e^{u} du = -e^{u}+c = -e^{\cos x}+c$

d) Here, $u = x^2+2x \Rightarrow du = (2x+2)dx = 2(x+1)dx \Rightarrow dx = \frac{du}{2(x+1)}$.

Therefore, $\int (x+1)\sqrt{x^2+2x}dx = \frac{1}{2}\int \sqrt{u}du = \frac{1}{3}u^{\frac{3}{2}}+c = \frac{1}{3}(x^2+2x)^{\frac{3}{2}}+c$

e) From the integral, $h(x)=2x^3+x$, $g(x)=x^4+x^2$, $f(x)=x^{49}$

Besides, $h(x)=\frac{1}{2}g'(x)=2x^3+x$.

So, let $u=x^4+x^2 \Rightarrow du = 2(2x^3+x)dx$.

Thus, $\int (2x^3+x)(x^4+x^2)^{49}dx = \frac{1}{2}\int u^{49}du = \frac{u^{50}}{100}+c = \frac{(x^4+x^2)^{50}}{100}+c$

f) Here, $u=e^x+1, du=e^x dx$.

Then, $\int e^x(e^x+1)^7 dx = \int u^7 du = \frac{1}{8}u^8+c = \frac{1}{8}(e^x+1)^8+c$

g) Look ! if we take $h(x)=\text{x}^3$, $f(x)=\frac{1}{x^2}$, $g(x)=4\text{x}^4+1$, the

integrand $I(x)=\frac{\text{x}^3}{(4\text{x}^4+1)^2}$ is of the form $\frac{h(x)}{f(g(x))}=\frac{\text{x}^3}{(4\text{x}^4+1)^2}$.

Besides, $h(x)=\text{x}^3$ resembles $g'(x)=16x^3$. That is $h(x)=\frac{1}{16}g'(x)$.

So, by the above hint, use the substitution, $u=g(x)=4x^4+1 \Rightarrow du=16x^3dx$.

Therefore, $\int \frac{\text{x}^3}{(4\text{x}^4+1)^2}dx = \frac{1}{16}\int \frac{1}{u^2}du = -\frac{1}{16u}+c = -\frac{1}{16(4\text{x}^4+1)}+c$.

h) Look ! if we take $h(x)=\sin 2x$, $f(x)=\frac{1}{x^3}$, $g(x)=5+\cos 2x$, the

integrand $I(x)=\frac{\sin 2x}{(5+\cos 2x)^3}$ is of the form $\frac{h(x)}{f(g(x))}=\frac{\sin 2x}{(5+\cos 2x)^3}$.

Besides, $h(x)=\sin 2x$ resembles $g'(x)=-2\sin 2x$. That is $h(x)=-\frac{1}{2}g'(x)$.

j) Use the substitution: $u=2x+\frac{1}{x}\Rightarrow du=(2-\frac{1}{x^2})dx$

So, $\int(2-\frac{1}{x^2})(2x+\frac{1}{x})dx=\int udu=\frac{u^2}{2}+c=\frac{1}{2}(2x+\frac{1}{x})^2+c$

k) Use the substitution: $u=x+\ln x\Rightarrow du=(1+\frac{1}{x})dx$

So, $\int(1+\frac{1}{x})\sin(x+\ln x)dx=\int\sin udu=-\cos u+c=-\cos(x+\ln x)+c$

Form-2: Integrals of the form $\int x^n f(x^r+a)dx$:

In such forms use the substitution $u=x^r+a$.

Then write x^n as $x^n=x^{n-r}.x^r$

Solve x^{n-r} in terms of u from $u=x^r+a$

Evaluate the integral.

Examples: Evaluate the integrals by applying the above procedures.

a) $\int x^3\sqrt{3+x^2}dx$ *b)* $\int x^5\sqrt{x^3+1}dx$ *c)* $\int x^8\sqrt{x^3+1}dx$ *d)* $\int x^9\sqrt[3]{x^5+1}dx$

e) $\int x^2(x-3)^5dx$ *f)* $\int x\sqrt{x+2}dx$ *g)* $\int\frac{x}{\sqrt{x+1}}dx$ *h)* $\int\frac{x^3}{\sqrt{x^2+4}}dx$

Solution:

a) Let $u=3+x^2$, $du=2xdx\Rightarrow xdx=1/2du$.

Besides, $x^3=x^2.x$ and $u=3+x^2\Rightarrow x^2=u-3$.

Thus, $\int x^3\sqrt{3+x^2}dx=\int x^2.x\sqrt{3+x^2}dx=\frac{1}{2}\int(u-3)\sqrt{u}du$

$$=\frac{1}{2}\int(u^{3/2}-3\sqrt{u})du=\frac{1}{5}u^{5/2}-u^{3/2}+c$$

$$=1/5(3+x^2)^{5/2}-(3+x^2)^{3/2}+c$$

b) Here, $u = x^3+1$, $du = 3x^2dx, x^3 = u-1$.

Thus, $\int x^5\sqrt{x^3+1}dx = \int x^3.x^2\sqrt{x^3+1}dx = \frac{1}{3}\int (u-1)\sqrt{u}du$

$$= \frac{2}{15}u^{5/2} - \frac{2}{9}u^{3/2} + c = \frac{2}{15}(x^3+1)^{5/2} - \frac{2}{9}(x^3+1)^{3/2} + c$$

c) Here, $u = x^3+1 \Rightarrow du = 3x^2dx$, $x^3 = u-1$.

Then, $\int x^8\sqrt{x^3+1}dx = \int (x^3)^2.x^2\sqrt{x^3+1}dx = \frac{1}{3}\int (u-1)^2\sqrt{u}du$

$$= \frac{1}{3}[\frac{2}{7}u^{7/2} + \frac{4}{5}u^{5/2} + \frac{2}{3}u^{3/2}] + c$$

$$= \frac{2}{3}[\frac{1}{7}(x^3+1)^{7/2} + \frac{2}{5}(x^3+1)^{5/2} + \frac{1}{3}(x^3+1)^{3/2}]$$

d) Let $u = x^5+1 \Rightarrow du = 5x^4$, $x^5 = u-1$, $x^9 = x^5.x^4$.

$\int x^9\sqrt[3]{x^5+1}dx = \int x^5.x^4\sqrt[3]{x^5+1} = \frac{1}{5}\int (u-1)\sqrt[3]{u}du$

$$= \frac{1}{5}\int (u^{4/3} - u^{1/3})du = \frac{3}{35}u^{7/3} - \frac{3}{20}u^{4/3} + c$$

$$= \frac{3}{35}(x^5+1)^{7/3} - \frac{3}{20}(x^5+1)^{4/3} + c$$

e) Here, $u = x-3$, $dx = du$, $x = u+3$. Then,

$\int x^2(x-3)^5dx = \int (u+3)^2u^5du = \int (u^2+6u+9)u^5du$

$$= \int (u^7 + 6u^6 + 9u^5)du$$

$$= \tfrac{1}{8}u^8 + \tfrac{6}{7}u^7 + \tfrac{3}{2}u^6 + c$$

$$= \tfrac{1}{8}(x-3)^8 + \tfrac{6}{7}(x-3)^7 + \tfrac{3}{2}(x-3)^6 + c$$

f) Here, $u = \sqrt{x+2}$, $x = u^2-2$, $dx = 2\sqrt{x+2}du = 2udu$. Then,

$\int x\sqrt{x+2}dx = 2\int (u^2-2)u^2du = 2\int (u^4-2u^2)du$

$$= \tfrac{2}{5}u^5 - \tfrac{4}{3}u^3 + c$$

$$= \tfrac{2}{5}(\sqrt{x+2})^5 - \tfrac{4}{3}(\sqrt{x+2})^3 + c$$

g) Here, let $u = x+1, du = dx, x = u-1$.

Hence, $\int \frac{x}{\sqrt{x+1}}dx = \int \frac{u-1}{\sqrt{u}}du = \int (\sqrt{u} - \frac{1}{\sqrt{u}})du$

$$= \frac{2}{3}u^{3/2} - 2\sqrt{u} + c = \frac{2}{3}(x+1)^{3/2} - 2\sqrt{x+1} + c$$

h) Here, let $u = x^2+4, du = 2xdx, x^2 = u-4$.

Hence, $\int \frac{x^3}{\sqrt{x^2+4}}dx = \frac{1}{2}\int \frac{u-4}{\sqrt{u}}du = \frac{1}{2}\int (\sqrt{u} - \frac{4}{\sqrt{u}})du$

$$= \frac{1}{3}u^{3/2} - 4\sqrt{u} + c = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + c$$

Form-3: Integrals of the form $\int f(a+\sqrt{x})dx, \int f(\frac{a}{x})dx$:

In such forms use the substitution $u = a+\sqrt{x}$ OR, $u = \frac{a}{x}$.

Examples: Evaluate the following integrals.

a) $\int (1+\sqrt{x})^8 dx$ *b)* $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$ *c)* $\int \frac{(1+\sqrt{x})^4}{\sqrt{x}}dx$ *d)* $\int \frac{dx}{(1+\sqrt{x})^4}$

e) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx$ *f)* $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}dx$ *g)* $\int \frac{1}{x^2}\sin(\frac{5}{x})dx$ *h)* $\int \frac{e^{\frac{1}{x}}}{x^2}dx$

Solution:

a) Using the above hint, $u = 1+\sqrt{x}, \sqrt{x} = u-1, dx = 2\sqrt{x}du = 2(u-1)du$.

Therefore, $\int (1+\sqrt{x})^8 dx = 2\int u^8(u-1)du = 2\int (u^9 - u^8)du$

$$= 2(\frac{1}{10}u^{10} - \frac{1}{9}u^9) = \frac{1}{5}(1+\sqrt{x})^{10} - \frac{2}{9}(1+\sqrt{x})^9 + c$$

b) Here, $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{1}{\sqrt{x}}dx = 2du$.

Thus, $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = 2\int e^u du = 2e^u + c = 2e^{\sqrt{x}} + c$.

c) Let $u = 1+\sqrt{x}, du = \frac{1}{2\sqrt{x}}dx \Rightarrow \frac{1}{\sqrt{x}}dx = 2du.$

Hence, $\int \frac{(1+\sqrt{x})^4}{\sqrt{x}}dx = 2\int u^4 du = \frac{u^5}{5} + c = \frac{2(1+\sqrt{x})^5}{5} + c.$

d) Let $u = 1+\sqrt{x}, \sqrt{x} = u-1,\ du = \frac{1}{2\sqrt{x}}dx \Rightarrow dx = 2\sqrt{x}du = 2(u-1).$

Hence, $\int \frac{1}{(1+\sqrt{x})^4}dx = \int \frac{2(u-1)}{u^4}du = 2\int (\frac{1}{u^3} - \frac{1}{u^4})du$

$$= \frac{2}{3u^3} - \frac{1}{u^2} + c = \frac{2}{3(1+\sqrt{x})^3} - \frac{1}{(1+\sqrt{x})^2} + c$$

e) $\int \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int \sin u du = -2\cos u + c = -2\cos\sqrt{x} + c$

f) $\int \frac{\sec^2\sqrt{x}}{\sqrt{x}}dx = 2\int \sec^2 u du = 2\tan u + c = 2\tan\sqrt{x} + c$

g) Let $u = \frac{5}{x}, du = -\frac{5}{x^2}dx \Rightarrow dx = -\frac{x^2}{5}du.$

Hence, $\int \frac{1}{x^2}\sin(\frac{5}{x})dx = -\frac{1}{5}\int \sin u du = \frac{1}{5}\cos u + c = \frac{1}{5}\cos(\frac{5}{x}) + c$

h) Let $u = \frac{1}{x}, du = -\frac{1}{x^2}dx \Rightarrow dx = -x^2 du$

$$\Rightarrow \int \frac{e^{\frac{1}{x}}}{x^2}dx = -\int e^u du = e^u + c = -e^{\frac{1}{x}} + c$$

Form-4: Integrals of the form $\int \frac{1}{x} f(a+\ln x)dx$ **:**

In such forms use the substitution $u = a + \ln x$.

Examples: Evaluate the following integrals.

a) $\int \frac{\cos(\ln x)dx}{x}$ b) $\int \frac{\tan(\ln x)}{x}dx$ c) $\int \frac{\sqrt{\ln x}}{x}dx$ d) $\int \frac{(\ln x)^5}{x}dx$

e) $\int \frac{(1+\ln x)^{19}}{x}dx$ f) $\int \frac{1}{x(3+\ln x)}dx$ g) $\int \cos(x+\ln x)(1+\frac{1}{x})dx$

Solution:

a) Here, $u = \ln x, du = 1/xdx.$

Then, $\int \frac{\sin(\ln x)}{x}dx = \int \sin u du = -\cos u + c = -\cos(\ln x) + c.$

b) Here, $u = \ln x, du = \frac{1}{x}dx.$

Then, $\int \frac{\tan(\ln x)}{x}dx = \int \tan u du = -\ln|\cos u| + c = -\ln|\cos(\ln x)| + c.$

c) Here, $u = \ln x, du = \frac{1}{x}dx.$ Then, $\int \frac{\sqrt{\ln x}}{x}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}(\ln x)^{\frac{3}{2}} + c$.

d) Here, $u = \ln x, du = \frac{1}{x}dx \Rightarrow \int \frac{(\ln x)^5}{x}dx = \int u^5 dx = \frac{u^6}{6} + c = \frac{(\ln x)^5}{6} + c$

e) Here, $u = 1 + \ln x, du = \frac{1}{x}dx.$

Therefore, $\int \frac{(1+\ln x)^{19}}{x}dx = \int u^{19}du = \frac{u^{20}}{20} + c = \frac{(1+\ln x)^{20}}{20} + c$

f) Let $u = 3 + \ln x, du = \frac{1}{x}dx \Rightarrow \int \frac{1}{x(1+\ln x)}dx = \int \frac{1}{u}du = \ln|u| + c = \ln|3+\ln x| + c$

g) Here, $u = x + \ln x, du = (1+\frac{1}{x})dx.$

Therefore, $\int \cos(x+\ln x)(1+\frac{1}{x})dx = \int \cos u du = \sin u + c = \sin(x+\ln x) + c.$

Form-5: When part of the integrand contains inverse trigonometric function like $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$, substitute the inverse functions.

Examples: Evaluate the following integrals

$a)\int \frac{dx}{(1+x^2)\tan^{-1}x}$ $b)\int \frac{(\sin^{-1}2x)^2dx}{\sqrt{1-4x^2}}$ $c)\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$ $d)\int \frac{(\sec^{-1}x)^3}{x\sqrt{x^2-1}}dx$

Solution:

a) By substitution with $u=\tan^{-1}x$, $du=\frac{1}{1+x^2}dx$.

Hence, $\int \frac{1}{(1+x^2)\tan^{-1}x}dx=\int \frac{du}{u}=\ln|u|+c=\ln|\tan^{-1}x|+c$

b) By substitution with $u=\sin^{-1}2x, du=\frac{2dx}{\sqrt{1-4x^2}}$.

Hence, $\int \frac{(\sin^{-1}2x)^2dx}{\sqrt{1-4x^2}}=\int \frac{u^2}{2}du=\frac{1}{6}u^3+c=\frac{1}{6}(\sin^{-1}2x)^3+c$

c) By substitution with $u=\tan^{-1}x$, $du=\frac{1}{x^2+1}dx$.

Hence, $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx=\int e^u du=e^u+c=e^{\tan^{-1}x}+c$

d) By substituion $u=\sec^{-1}x \Rightarrow du=\frac{dx}{x\sqrt{x^2-1}}$.

Then, $\int \frac{(\sec^{-1}x)^3}{x\sqrt{x^2-1}}dx=\int u^3du=\frac{u^4}{4}+c=\frac{1}{4}(\sec^{-1}x)^4+c$

6.2.2 Integration by Parts

Let f and g be differentiable functions. Then, from the product rule of differentiation, we have that $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$.

Now, by integrating both sides, we get

$$\int (fg)'(x)dx = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

$$\Rightarrow (fg)(x) = \int f'(x)g(x)dx + \int f(x)g'(x)dx$$

Thus, $\int f'(x)g(x)dx = (fg)(x) - \int f(x)g'(x)dx$

OR

$$\int f(x)g'(x)dx = (fg)(x) - \int f'(x)g(x)dx$$

This method is useful to evaluate integrals involving product of algebraic and transcendental functions as integrand.

It is a method of finding integrals by splitting the integrand into two parts, one $f(x)$ and the other $g'(x)$. *As a result, the method is called integration by parts.* Here, $f(x)$ and $g'(x)$ are chosen in such away that $f(x)$ is easy to differentiate, $g(x)$ is easily obtained from $g'(x)$ and $\int f'(x)g(x)dx$ is easier to integrate. There is no hard and fast rule on how to choose $f(x)$ and $g'(x)$. It is through practice and experience as well as trial and error that one can acquire the skill. Sometimes, we can use the following rule as a hint.

This rule can be memorized as **LIATE**-Rules of integration.

L = Logarthmic Functions: $\ln f(x), \log_a f(x)$

I = Inverse Functions: $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x, \sec^{-1} x$, and others

A = Algebraic Functions: $f(x) = p(x)$ (Some polynomial), $f(x) = \dfrac{p(x)}{q(x)}$

T = Trigonometric Functions: $\sin x, \cos x, \tan x, \sec x, \cot x, \csc x$

E = Exponential Functions: $f(x) = e^{kx}$, $f(x) = a^{kx}$

How to use LIATE rule? This rule is used as follow:

First: Identify two functions in the integrand such that they are in one of the above function categories.

Second: Take as $f(x)$ the one that belongs to the category you will get **FIRST** in going from top to down and as $g'(x)$ the one that belongs to a category below the category of f. For instance, consider an integral $\int x^3e^x dx$. In this integral, the two functions are x^3 (which is algebraic since it is a polynomial) and e^x (which is exponential). But if you see in the **LIATE** rule, from top to down, you will get algebraic categories first and exponential categories next (below algebraic categories). So, it is a must that you will use $f(x)=x^3$, $g'(x)=e^x$.

Examples:

1. Evaluate the following integrals.

a) $\int x\sin xdx$ b) $\int x\cos 3xdx$ c) $\int xe^x dx$ d) $\int x^2e^x dx$

e) $\int x.2^x dx$ f) $\int x^2 \sin xdx$ g) $\int x^2 \cos xdx$

Solution:

a) Here, the integrand $I(x)=x\sin x$ is the product of a polynomial which is Algebraic (**A**) and trigonometric which is transcendental (**T**). So, by **LIATE** rule, choose $f(x)=x,\ g'(x)=\sin x$ such that $f'(x)=1,\ g(x)=-\cos x$.

Thus, $\int x\sin xdx=-x\cos x-\int -\cos xdx=-x\cos x+\sin x+c$.

b) Using similar reasoning as part (a), we can choose

$f(x)=x,\ g'(x)=\cos 3x$ such that $f'(x)=1,\ g(x)=\dfrac{\sin 3x}{3}$.

Thus, $\int x\cos 3xdx=\dfrac{x\sin 3x}{3}-\int\dfrac{\sin 3x}{3}dx=\dfrac{x\sin 3x}{3}+\dfrac{\cos 3x}{9}+c$.

c) Here, the integral is of the form $\int p(x)e^{ax}dx$. So, choose

$f(x)=x,\ g'(x)=e^x$ such that $f'(x)=1,\ g(x)=e^x$.

Thus, $\int xe^x dx=xe^x-\int e^x dx=xe^x-e^x+c$

d) Choose $f(x)=x^2,\ g'(x)=e^x$ such that $f'(x)=2x,\ g(x)=e^x$. Thus,

$\int x^2e^xdx = x^2e^x - 2\int xe^xd$. But from part (a), $\int xe^xdx = xe^x - e^x + c$.

Thus, $\int x^2e^xdx = x^2e^x - 2\int xe^xdx = x^2e^x - 2xe^x + 2e^x + c$

e) Here, the integral is of the form $\int p(x)a^xdx$. So, choose

$$f(x)=x,\ g'(x)=2^x \text{ such that } f'(x)=1,\ g(x)=\frac{2^x}{\ln 2}.$$

$$\text{Thus, } \int x2^xdx = x\frac{2^x}{\ln 2} - \int\frac{2^x}{\ln 2}dx = x\frac{2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2} + c$$

f) Let $f(x)=x^2, g'(x)=\sin x \Rightarrow f'(x)=2x, g(x)=-\cos x$.

Hence, $\int x^2\sin xdx = -x^2\cos x + 2\int x\cos xdx$.

But using integration by parts again, $2\int x\cos xdx = 2x\sin x + 2\cos x$.

Hence, $\int x^2\sin xdx = -x^2\cos x + 2x\sin x + 2\cos x + c$.

g) Let $f(x)=x^2, g'(x)=\cos x$ and note that $\int x\sin xdx = -x\cos x + \sin x$

Hence, $\int x^2\cos xdx = x^2\sin x - 2\int x\sin xdx = x^2\sin x - 2(-x\cos x + \sin x)$

2. Evaluate the following integrals

a) $\int \ln xdx$ b) $\int x\ln xdx$ c) $\int x^2\ln xdx$ d) $\int x^3\ln xdx$

e) $\int(\ln x)^2dx$ f) $\int\sqrt{x}\ln xdx$ g) $\int(3x^2+2x)\ln xdx$

Solution: For the form $\int p(x)\ln xdx$, choose $f(x)=\ln x,\ g'(x)=p(x)$

a) $f(x)=\ln x,\ g'(x)=1 \Rightarrow f'(x)=\frac{1}{x},\ g(x)=x.$

Thus, $\int \ln xdx = x\ln x - \int 1dx = x\ln x - x + c$

b) $f(x)=\ln x,\ g'(x)=x \Rightarrow f'(x)=\frac{1}{x},\ g(x)=\frac{x^2}{2}.$

$$\text{Thus, } \int x\ln xdx = \frac{x^2}{2}\ln x - \int\frac{x}{2}dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

c) $f(x)=\ln x,\ g'(x)=x^2 \Rightarrow f'(x)=\frac{1}{x},\ g(x)=\frac{x^3}{3}.$

Thus, $\int x^2 \ln x dx=\frac{x^3}{3}\ln x-\int\frac{x^2}{3}dx=\frac{x^3}{3}\ln x-\frac{x^3}{9}+c$

d) $f(x)=\ln x,\ g'(x)=x^3 \Rightarrow f'(x)=\frac{1}{x},\ g(x)=\frac{x^4}{4}.$

Thus, $\int x^3 \ln x dx=\frac{x^4}{4}\ln x-\int\frac{x^3}{4}dx=\frac{x^4}{4}\ln x-\frac{x^4}{16}+c$

e) $f(x)=(\ln x)^2, g'(x)=1 \Rightarrow f'(x)=\frac{2\ln x}{x}, g(x)=x.$

Thus, $\int(\ln x)^2dx=x(\ln x)^2-2\int \ln x dx=x(\ln x)^2-2x\ln x+2x+c$

f) Here, let $f(x)=\ln x,\ g'(x)=\sqrt{x} \Rightarrow f'(x)=\frac{1}{x}, g(x)=\frac{2}{3}x^{\frac{3}{2}}$

Thus, $\int\sqrt{x}\ln x dx=\frac{2}{3}x^{\frac{3}{2}}\ln x-\int\frac{2}{3}\sqrt{x}dx=\frac{2}{3}x^{\frac{3}{2}}\ln x-\frac{4}{9}x^{\frac{3}{2}}+c$

g) $f(x)=\ln x,\ g'(x)=3x^2+2x \Rightarrow f'(x)=\frac{1}{x},\ g(x)=x^3+x^2.$ Thus,

$\int(3x^2+2x)\ln x dx=(x^3+x^2)\ln x-\int(x^2+x)dx=(x^3+x^2)\ln x-(\frac{x^3}{3}+\frac{x^2}{2})$

3. Evaluate the following integrals.

a) $\int\sin^{-1}x dx$ b) $\int\cos^{-1}x dx$ c) $\int x\sec^{-1}x dx$

d) $\int x^2\tan^{-1}x dx$ e) $\int\frac{1}{x}\sin^{-1}(\ln x)dx$ f) $\int\tan^{-1}x dx$

Solution:

a) Let $f(x)=\sin^{-1}x, g'(x)=1 \Rightarrow f'(x)=\frac{1}{\sqrt{1-x^2}}, g(x)=x.$

Hence, $\int\sin^{-1}x dx=x\sin^{-1}x-\int\frac{x}{\sqrt{1-x^2}}dx$

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But using the substitution, $u=\sqrt{1-x^2} \Rightarrow dx=\frac{-\sqrt{1-x^2}}{x}du.$

We have $\int \frac{x}{\sqrt{1-x^2}}dx=-\int du=-u=-\sqrt{1-x^2}.$

Therefore, $\int \sin^{-1}xdx=x\sin^{-1}x-\int \frac{x}{\sqrt{1-x^2}}dx=x\sin^{-1}x+\sqrt{1-x^2}+c$

b) Let $f(x)=\cos^{-1}x, g'(x)=1 \Rightarrow f'(x)=\frac{-1}{\sqrt{1-x^2}}, g(x)=x.$

Hence, $\int \cos^{-1}xdx=x\cos^{-1}x+\int \frac{x}{\sqrt{1-x^2}}dx.$

But using the substitution, $u=\sqrt{1-x^2}, \int \frac{x}{\sqrt{1-x^2}}dx=-\int du=-u=-\sqrt{1-x^2}.$

Therefore, $\int \cos^{-1}xdx=x\cos^{-1}x-\int \frac{x}{\sqrt{1-x^2}}dx=x\cos^{-1}x-\sqrt{1-x^2}+c.$

c) Let $f(x)=\sec^{-1}x, g'(x)=x \Rightarrow f'(x)=\frac{1}{x\sqrt{x^2-1}}, g(x)=\frac{x^2}{2}.$

Hence, $\int x\sec^{-1}xdx=\frac{x^2}{2}\sec^{-1}x-\frac{1}{2}\int \frac{x}{\sqrt{x^2-1}}dx$

But using $u=\sqrt{x^2-1} \Rightarrow du=\frac{x}{\sqrt{x^2-1}}dx \Rightarrow \int \frac{x}{\sqrt{x^2-1}}dx=\int du=u=\sqrt{x^2-1}.$

Therefore, $\int x\sec^{-1}xdx=\frac{x^2}{2}\sec^{-1}x-\frac{1}{2}\int \frac{x}{\sqrt{x^2-1}}dx=\frac{x^2}{2}\sec^{-1}x-\frac{\sqrt{x^2-1}}{2}+c$

d) Let $f(x)=\tan^{-1}x, g'(x)=x^2 \Rightarrow f'(x)=\frac{1}{1+x^2}, g(x)=\frac{x^3}{3}.$

So, $\int x^2\tan^{-1}xdx=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\int \frac{x^3dx}{1+x^2}=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\int \left(x-\frac{x}{1+x^2}\right)dx$

$$=\frac{x^3\tan^{-1}x}{3}-\frac{1}{3}\left(\frac{1}{2}x^2-\frac{1}{2}\ln(1+x^2)\right)+c$$

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e) Here, $u=\ln x$, $du=\frac{1}{x}dx \Rightarrow dx = xdu$. Thus,

$$\int \frac{1}{x}\sin^{-1}(\ln x)dx=\int \sin^{-1} u du = u\sin^{-1} u+\sqrt{1-u^2}=(\ln x)\sin^{-1}(\ln x)+\sqrt{1-(\ln x)^2}$$

f) $\int \tan^{-1} x dx = x\tan^{-1} x - \int \frac{x}{1+x^2}dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2)+c.$

4. Evaluate the following integrals.

$a)\int e^{\sqrt{x}}dx$ $\quad b)\int \cos\sqrt{x}dx$ $\quad c)\int \cos x\ln(\sin x)dx$ $\quad d)\int x\cos^2 xdx$

$e)\int x\sec^2 xdx$ $\quad f)\int x\csc^2 xdx$ $\quad g)\int \cos^3 x\ln(\sin x)dx$

Solution:

a) Using the substitution,

$$u=\sqrt{x}, du=\frac{1}{2\sqrt{x}}dx \Rightarrow \int e^{\sqrt{x}}dx=\int 2ue^u du = 2ue^u - e^u + c = 2\sqrt{x}e^{\sqrt{x}} - e^{\sqrt{x}} + c.$$

b) Using the substitution,

$$u=\sqrt{x}, du=\frac{1}{2\sqrt{x}}dx \Rightarrow \int \cos\sqrt{x}dx=\int 2u\cos udu = 2u\sin u - 2\int \sin udu + c$$

$$= 2u\sin u + 2\cos u + c = 2\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x} + c.$$

c) Using the substitution, $u=\sin x, du=\cos xdx$, we have

$$\int \cos x\ln(\sin x)dx=\int \ln udu = u\ln u - u + c = \sin x\ln(\sin x) - \sin x + c$$

d) Let $f(x)=x, g'(x)=\cos^2 x, f'(x)=1$. Then,

$$g(x)=\int \cos^2 xdx=\int \frac{1+\cos 2x}{2}dx=\frac{1}{2}x+\frac{1}{4}\sin 2x$$

Hence, $\int x\cos^2 xdx=\frac{1}{4}x^2+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x+c$

e) Here, let $f(x)=x$, $g'(x)=\sec^2 x \Rightarrow f'(x)=1$, $g(x)=\tan x$.

Thus, $\int x\sec^2 xdx = x\tan x - \int \tan xdx = x\tan x + \ln|\cos x| + c$.

f) Here, let $f(x)=x$, $g'(x)=\csc^2 x \Rightarrow f'(x)=1$, $g(x)=-\cot x$.

Thus, $\int x\csc^2 xdx = -x\cot x + \int \cot xdx = \ln|\sin x| - x\cot x + c$.

Remark (Repeated use of integration by parts):

Sometimes an integral may repeat itself when it is integrated. In such cases, the integral is evaluated by collecting like terms together.

Examples:

1. Evaluate the following integrals using the remark.

$a)\int e^x \sin x dx$ $\quad b)\int e^x \cos x dx$ $\quad c)\int 2^x \cos x dx$ $\quad d)\int \cos(\ln x) dx$

Solution: a) Here, $f(x)=e^x$, $g'(x)=\sin x \Rightarrow f'(x)=e^x$, $g(x)=-\cos x$

Thus, $\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$.................(*).

Again consider $\int e^x \cos x dx$.

Here, also let $f(x)=e^x$, $g'(x)=\cos x \Rightarrow f'(x)=e^x$, $g(x)=\sin x$

Then, $\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$.

Put this result in (*) in the place of $\int e^x \cos x dx$.

Thus, $\int e^x \sin x dx = -e^x \cos x + \int e^x \cos x dx$

$$= -e^x \cos x + e^x \sin x - \int e^x \sin x dx$$

$$\Rightarrow 2\int e^x \sin x dx = e^x \sin x - e^x \cos x$$

$$\Rightarrow \int e^x \sin x dx = 1/2(e^x \sin x - e^x \cos x) + c$$

Therefore, $\int e^x \sin x dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + c$.

b) Similarly as we did in part (a), $\int e^x \cos x dx = \frac{1}{2}(e^x \sin x + e^x \cos x) + c$

c) $f(x)=2^x$, $g'(x)=\cos x \Rightarrow f'(x)=(\ln 2).2^x$, $g(x)=\sin x$

Thus, $\int 2^x \cos x dx = 2^x \sin x - \int (\ln 2) 2^x \sin x dx$

$$= 2^x \sin x - (\ln 2)\int 2^x \sin x dx.................(*)$$

Again consider $\int 2^x \sin x dx$. Here, also let

$f(x)=2^x,\ g'(x)=\sin x \Rightarrow f'(x)=(\ln 2)2^x,\ g(x)=-\cos x$

Then, $\int 2^x \sin x dx = -2^x \cos x + \int (\ln 2)2^x \cos x dx$.

Put this result in $(*)$ in the place of the integral $\int 2^x \sin x dx$.

$$\int 2^x \cos x dx = 2^x \sin x - (\ln 2)\int 2^x \sin x dx$$

$$= 2^x \sin x - (\ln 2)[-2^x \cos x + \int (\ln 2)2^x \cos x dx]$$

$$\Rightarrow \int 2^x \cos x dx = 2^x \sin x + (\ln 2)2^x \cos x - (\ln 2)^2 \int 2^x \cos x dx$$

$$\Rightarrow (1+(\ln 2)^2)\int 2^x \cos x dx = 2^x(\sin x + (\ln 2)\cos x)$$

$$\Rightarrow \int 2^x \cos x dx = \frac{2^x}{(1+(\ln 2)^2)}(\sin x + (\ln 2)\cos x) + c$$

d) Here, let $f'(x)=1,\ g(x)=\cos(\ln x) \Rightarrow f(x)=x,\ g'(x)=-\frac{\sin(\ln x)}{x}$.

Thus, $\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$..................(*)

Again consider $\int \sin(\ln x)dx$. Here, also let

$f'(x)=1,\ g(x)=\sin(\ln x) \Rightarrow f(x)=x,\ g'(x)=\frac{\cos(\ln x)}{x}$

Then, $\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$. Put this result in $(*)$.

Thus, $\int \cos(\ln x)dx = x\cos(\ln x) + \int \sin(\ln x)dx$

$$= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)dx$$

$$\Rightarrow 2\int \cos(\ln x)dx = x\cos(\ln x) + x\sin(\ln x) + c$$

$$\Rightarrow \int \cos(\ln x)dx = \frac{1}{2}(x\cos(\ln x) + x\sin(\ln x)) + c$$

BM Generalized Rule: For integrals of the form $\int x^n e^{x^m} dx$.

For integrals of the form $\int x^n e^{x^m} dx, n > m \geq 1, m, n \in N$, choose $f(x) = x^{n-m+1}$ and $g'(x) = x^{m-1} e^{x^m}$.

Examples: Evaluate the following integrals using the remark

a) $\int x^5 e^{x^3} dx$ b) $\int x^3 e^{x^2} dx$ c) $\int x^5 e^{x^2} dx$

Solution: Here, all the integrals are of the form $\int x^n e^{x^m} dx$. So, to evaluate each of these integrals, apply the BM Generalized rule.

a) Here, $n = 5$, $m = 3$. So, by BM generalized rule, we can choose $f(x) = x^{n-m+1} = x^{5-3+1} = x^3, g'(x) = x^{m-1} e^{x^m} = x^{3-1} e^{x^3} = x^2 e^{x^3}$. For these choices, we have $f'(x) = 3x^2$, $g(x) = \frac{e^{x^3}}{3}$. Thus, using integration by parts, the integral becomes $\int x^5 e^{x^3} dx = \frac{x^3 e^{x^3}}{3} - \int x^2 e^{x^3} dx = \frac{x^3 e^{x^3}}{3} - \frac{e^{x^3}}{3} + c$.

b) Choose $f(x) = x^2$, $g'(x) = xe^{x^2}$ such that $f'(x) = 2x$, $g(x) = \frac{e^{x^2}}{2}$.

Thus, $\int x^3 e^{x^2} dx = \frac{x^2 e^{x^2}}{2} - \int xe^{x^2} = \frac{x^2 e^{x^2}}{2} - \frac{e^{x^2}}{2} + c$.

c) Choose $f(x) = x^4$, $g'(x) = xe^{x^2}$ such that $f'(x) = 4x^3$, $g(x) = \frac{e^{x^2}}{2}$.

Thus, $\int x^5 e^{x^2} dx = \frac{x^4 e^{x^2}}{2} - 2\int x^3 e^{x^2} = \frac{x^4 e^{x^2}}{2} - x^2 e^{x^2} + e^{x^2} + c$.

6.2.3 Trigonometric Integrals

Here, we are going to discuss how to integrate products or powers of the six trigonometric functions $\cos\theta, \sin\theta, \tan\theta, \sec\theta, \csc\theta$ and $\cot\theta$. Any integral that involves these functions, their powers, products and any combinations is called *trigonometric integral*.

Different Forms of Trigonometric Integrals

I) Integrals of Powers of Trigonometric: Reduction Formula

It is common to encounter integrals involving integrands with higher powers. In such cases, if it is possible to express the integral in terms of an integral with lower power of the same function, the resulting formula is known as reduction formula. The basic idea of the reduction formula is to change the higher powers of the integrand into lower power and then to use any known methods.

$$i)\int \sin^n x dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

$$ii)\int \cos^n x dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x dx$$

$$iii\int \tan^n x dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x dx$$

$$iv)\int \cot^n x dx = -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x dx$$

$$v)\int \sec^n x dx = \frac{\sec^{n-2} x\tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx$$

$$vi)\int \csc^n x dx = -\frac{\csc^{n-2} x\cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x dx$$

Examples: Using these reduction formula, evaluate the following integrals.

$a)\int \sin^4 x dx$ $\quad b)\int \cos^4 x dx$ $\quad c)\int \sec^4 x dx$ $\quad d)\int \tan^4 x dx$

$e)\int \sin^3 x dx$ $\quad f)\int \cos^3 x dx$ $\quad g)\int \tan^3 x dx$ $\quad h)\int \sec^3 x dx$

Solution:

$$a)\int \sin^4 x dx = -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\int \sin^2 x dx$$

$$= -\frac{\sin^3 x\cos x}{4} + \frac{3}{4}\left(-\frac{\sin x\cos x}{2} + \frac{1}{2}\int dx\right)$$

$$= -\frac{\sin^3 x\cos x}{4} - \frac{3\sin x\cos x}{8} + \frac{3x}{8} + c$$

$b)\int \cos^4 xdx = \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\int \cos^2 xdx = \frac{\cos^3 x\sin x}{4} + \frac{3}{4}\left(\frac{\cos x\sin x}{2} + \frac{1}{2}\int dx\right)$

$$= \frac{\cos^3 x\sin x}{4} + \frac{3}{8}\cos x\sin x + \frac{3}{8}x + c$$

$c)\int \sec^4 xdx = \frac{\sec^2 x\tan x}{3} + \frac{2}{3}\int \sec^2 xdx = \frac{\sec^2 x\tan x}{3} + \frac{2}{3}\tan x + c$

$d)\int \tan^4 xdx = \frac{1}{3}\tan^3 x - \int \tan^2 xdx = \frac{1}{3}\tan^3 x - [\tan x - \int dx]$

$$= \frac{1}{3}\tan^3 x - \tan x + x + c = \frac{1}{3}\tan^3 x + \tan x + c$$

II) *Integrals of Trigonometric Products:*

The form $\int \sin ax\cos bxdx$, $\int \sin ax\sin bxdx$ and $\int \cos ax\cos bxdx$

An integral involving any of such forms can easily be evaluated by using *product to sum* formula of trigonometric functions.

Product to sum formula

i) $\sin ax\sin bx = \frac{1}{2}\left(\cos(a-b)x - \cos(a+b)x\right)$

ii) $\cos ax\cos bx = \frac{1}{2}\left(\cos(a-b)x + \cos(a+b)x\right)$

iii) $\sin ax\cos bx = \frac{1}{2}\left(\sin(a-b)x + \sin(a+b)x\right)$

Examples: Evaluate the following trigonometric integrals.

a) $\int \sin 5x\sin 2xdx$ *b)* $\int \cos 4x\cos 3xdx$ *c)* $\int \sin 6x\cos 2xdx$

Solution: Using the ***product-to-sum*** formula, we have

a) $\int \sin 5x\sin 2xdx = \int(\frac{1}{2}\cos 3x - \frac{1}{2}\cos 7x)dx = \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + c$

b) $\int \cos 4x\cos 3xdx = \int(\frac{1}{2}\cos 7x + \frac{1}{2}\cos x)dx = \frac{1}{14}\sin 7x + \frac{1}{2}\sin x + c$

c) $\int \sin 6x\cos 2xdx = \int(\frac{1}{2}\sin 4x + \frac{1}{2}\sin 8x)dx = -\frac{1}{8}\cos 4x - \frac{1}{16}\cos 8x + c$

III) Integrals of the form $\int \sin^m x \cos^n x dx$:

To solve such integrals, we use different cases based on the values of m and n .

Case-I: If m is an odd integer and n is any real number, use the substitution $u=\cos x$ and the identity $\sin^2 x = 1-\cos^2 x$ to evaluate the integral.

Examples: Evaluate the following integrals

$a)\int \sin x\cos^2 xdx$ $\quad b)\int \sin x\cos^{19} xdx$ $\quad c)\int \sin^3 x\cos^4 xdx$

$d)\int \sin^3 x\sqrt[3]{\cos x}dx$ $\quad e)\int \sin^5 x\cos^4 xdx$ $\quad f)\int \sin^3 x\cos^6 xdx$

Solution: Since m (the power of $\sin x$)is odd in all of the given integrals, we use the substitution $u=\cos x$, $du=-\sin xdx$, $\sin^2 x = 1-\cos^2 x$.

$$a)\ \int \sin x\cos^2 xdx = \int \cos^2 x\sin xdx = -\int u^2 du = -\frac{1}{3}u^3 + c = -\frac{\cos^3 x}{3} + c$$

$$b)\ \int \sin x\cos^{19} xdx = \int \cos^{19} x\sin xdx = -\int u^{19} du = \frac{1}{20}u^{20} + c = \frac{1}{20}\cos^{20} x + c$$

$$c)\ \int \sin^3 x\cos^4 xdx = \int \sin^2 x\sin x\cos^4 xdx = \int (1-\cos^2 x)\cos^4 x\sin xdx$$

$$= -\int (1-u^2)u^4 du = -\int (u^4 - u^6)du$$

$$= \frac{1}{7}u^7 - \frac{1}{5}u^5 + c = \frac{1}{7}\cos^7 x - \frac{1}{5}\cos^5 x + c$$

$$d)\ \int \sin^3 x\sqrt[3]{\cos x}dx = \int \sin x\sin^2 x\sqrt[3]{\cos x}dx = \int (1-\cos^2 x)\sqrt[3]{\cos x}\sin xdx$$

$$= -\int (1-u^2)\sqrt[3]{u}du = \frac{3}{10}(\cos x)^{10/3} - \frac{3}{4}(\cos x)^{4/3} + c$$

e) $\int \sin^5 x\cos^4 x dx = \int \sin^4 x\cos^4 x\sin x dx = \int (1-\cos^2 x)^2 \cos^4 x\sin x dx$

$$= -\int (1-u^2)^2 u^4 du = -\int (1-2u^2+u^4)u^4 du$$

$$= -\int (u^4 - 2u^6 + u^8)du = \frac{2}{7}u^7 - \frac{1}{5}u^5 - \frac{1}{9}u^9 + c$$

$$= \frac{2}{7}\cos^7 x - \frac{1}{5}\cos^5 x - \frac{1}{9}\cos^9 x + c$$

f) $\int \sin^3 x\cos^6 x dx = \int \sin^2 x\cos^6 x\sin x dx = \int (1-\cos^2 x)\cos^6 x\sin x dx$

$$= -\int (1-u^2)u^6 du = -\int (u^6 - u^8)du$$

$$= \frac{1}{9}u^9 - \frac{1}{7}u^7 + c = \frac{1}{9}\cos^9 x - \frac{1}{7}\cos^7 x + c$$

Case-II: If n is an odd integer and m is any real number, use the substitution $u = \sin x$ and the identity $\cos^2 x = 1 - \sin^2 x$ to evaluate the integral.

Examples: Evaluate the following integrals

a) $\int \sin^2 x\cos^3 x dx$ b) $\int \sin^4 x\cos^5 x dx$ c) $\int \sqrt{\sin x}\cos^3 x dx$

d) $\int \sin^6 x\cos^5 x dx$ e*) $\int \sin^{199} x\cos x dx$ f) $\int \sin^8 x\cos x dx$

Solution: Since n (the power of $\cos x$) is odd in all of the given integrals except part (e), we use the substitution $u = \sin x$, $du = \cos x dx$, $\cos^2 x = 1 - \sin^2 x$.

a) $\int \sin^2 x\cos^3 x dx = \int \sin^2 x(\cos^2 x)\cos x dx = \int \sin^2 x(1-\sin^2 x)\cos x dx$

$$= \int u^2(1-u^2)du = \int (u^2 - u^4)du$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + c = \frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + c$$

b) $\int \sin^4 x\cos^5 x dx = \int \sin^4 x(\cos^2 x)^2 \cos x dx = \int \sin^4 x(1-\sin^2 x)^2 \cos x dx$

$$= \int u^4(1-u^2)^2 du = \int (u^8 - 2u^6 + u^4)du$$

$$= \frac{1}{9}u^9 - \frac{2}{7}u^7 + \frac{1}{5}u^5 = \frac{1}{9}\sin^9 x - \frac{2}{7}\sin^7 x + \frac{1}{5}\sin^5 x + c$$

c) $\int \sqrt{\sin x}\cos^3 x dx = \int \sqrt{\sin x}\cos^2 x\cos x dx = \int \sqrt{u}(1-u^2)du = \int (\sqrt{u} - u^{\frac{5}{2}})du$

$$= \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{7}u^{\frac{7}{2}} + c = \frac{2}{3}\sin^{\frac{3}{2}} x - \frac{2}{7}\sin^{\frac{7}{2}} x + c$$

d) $\int \sin^6 x\cos^5 x dx = \int \sin^6 x(\cos^2 x)^2 \cos x dx = \int \sin^6 x(1-\sin^2 x)^2 \cos x dx$

$$= \int u^6(1-u^2)^2 du = \int (u^6 - 2u^8 + u^{10})du$$

$$= \frac{1}{7}u^7 - \frac{2}{9}u^9 + \frac{1}{10}u^{10} = \frac{1}{7}\sin^7 x - \frac{2}{9}\sin^9 x + \frac{1}{10}\sin^{10} x$$

e) Here, both $m = 199, n = 1$ $n = 1$ are odd. So, we can use the substitution either $u = \cos x$ or $u = \sin x$. But if we use $u = \sin x$, the integral becomes difficult to evaluate. In such cases substitute the one with largest power.

Hence, $\int \sin^{199} x\cos x dx = \int u^{199} du = \frac{1}{200}u^{200} + c = \frac{1}{200}\sin^{200} x + c$.

f) $\int \sin^8 x\cos x dx = \int u^8 du = \frac{1}{9}u^9 + c = \frac{1}{9}\sin^9 x + c$

Case-III: If both m and n are even integers, then reduce the integrand into simpler integrand using half angle formula.

Half angle formula: $\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$

Examples: Evaluate the following integrals

a) $\int \sin^2 x\cos^2 x dx$ b) $\int \sin^4 x\cos^4 x dx$ c) $\int \sin^4 x\cos^6 x dx$

Solution:

a) Here, $m = n = 2$ which are both even. So, we use

$\cos^2 x = \frac{1+\cos 2x}{2}$, $\sin^2 x = \frac{1-\cos 2x}{2}$ to simplify the integrand.

That is $\int \sin^2 x\cos^2 x dx = \int (\frac{1-\cos 2x}{2})(\frac{1+\cos 2x}{2})dx = \frac{1}{4}\int (1-\cos^2 2x)dx$

$$= \frac{1}{4}\int (1 - (\frac{1+\cos 4x}{2})dx \text{ (Using } \cos^2 2x = \frac{1+\cos 4x}{2})$$

$$= \frac{1}{8}\int (1-\cos 4x)dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

b) Here, $m=4$, $n=4$ are even. So, use $\cos^2 x=\frac{1+\cos 2x}{2}, \sin^2 x=\frac{1-\cos 2x}{2}$

$$\int \sin^4 x\cos^4 x dx=\int(\sin^2 x)^2(\cos^2 x)^2 dx=\int(\frac{1-\cos 2x}{2})^2(\frac{1+\cos 2x}{2})^2 dx$$

$$=\frac{1}{4}\int(1-\cos^2 2x)^2 dx=\frac{1}{4}\int\left(1-(\frac{1+\cos 4x}{2})\right)^2 dx$$

$$=\frac{1}{4}\int\left(\frac{1}{2}-\frac{\cos 4x}{2}\right)^2 dx=\frac{1}{4}\int\left(\frac{1}{4}-\frac{\cos 4x}{2}+\frac{\cos^2 4x}{4}\right)dx$$

$$=\frac{1}{4}\int\left(\frac{1}{4}-\frac{\cos 4x}{2}+\frac{1+\cos 8x}{8}\right)dx=\frac{3}{32}x-\frac{1}{32}\sin 4x+\frac{1}{256}\sin 8x$$

IVI) Integrals of the form $\int \tan^m x\sec^n x dx$:

These forms of integrals are also performed using different cases.

Case-I: If m is an odd integer and n is any real number, use the substitution $u=\sec x$ and the identity $\tan^2 x=\sec^2 x-1$ to evaluate the integral.

Examples: Evaluate the following integrals

a) $\int \tan x\sec x\,dx$ *b)* $\int \tan x\sec^{99} x\,dx$ *c)* $\int \tan^3 x\sec^5 x\,dx$

d) $\int \tan x\sec^{\frac{3}{2}} x\,dx$ *e)* $\int \tan^5 x\sec^3 x\,dx$ *f)* $\int \tan^7 x\sec^5 x\,dx$

Solution: Since m (the power of $\tan x$)is odd in all of the given integrals, we use the substitution $u=\sec x$, $du=\sec x\tan x dx$, $\tan^2 x=\sec^2 x-1$.

a) $\int \tan x\sec x\,dx=\int du=u+c=\sec x+c$

b) $\int \tan x\sec^{99} x\,dx=\int u^{98}du=\frac{1}{99}u^{99}+c=\frac{1}{99}\sec^{99} x+c$

c) $\int \tan^3 x\sec^5 x\,dx=\int(\sec^2 x-1)\sec^4 x(\sec x\tan x)dx$

$$=\int(u^6-u^4)du=\frac{1}{7}u^7-\frac{1}{5}u^5=\frac{1}{7}\sec^7 x-\frac{1}{5}\sec^5 x+c$$

d) $\int \tan x\sec^{3/2} x\,dx=\int\sqrt{\sec x}(\sec x\tan x)dx=\int\sqrt{u}du=\frac{2}{3}u^{3/2}+c=\frac{2}{3}\sec^{3/2} x$

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e) $\int \tan^5 x\sec^3 x\,dx = \int \tan^4 x\sec^2 x(\sec x\tan x)dx$

$$= \int (\sec^2 x-1)^2 \sec^2 x(\sec x\tan x)dx$$

$$= \int (u^6 - 2u^4 + u^2)du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3$$

$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + c$$

f) $\int \tan^7 x\sec^5 x\,dx = \int \tan^6 x\sec^4 x(\sec x\tan x)dx$

$$= \int (\sec^2 x-1)^3 \sec^4 x(\sec x\tan x)dx$$

$$= \int (u^2-1)^3 u^4 du = \int (u^6 - 3u^4 + 3u^2 - 1)u^4 du$$

$$= \frac{1}{11}\sec^{11} x - \frac{1}{3}\sec^9 x + \frac{3}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c$$

Case-II: If n is an even integer and m is any real number, use the substitution $u = \tan x$ and the identity $\sec^2 x = 1+\tan^2 x$ to evaluate the integral.

Examples: Evaluate the following integrals

a) $\int \tan^2 x\sec^4 x dx$ *b)* $\int \tan^{12} x\sec^4 x dx$ *c)* $\int \tan^4 x\sec^6 x dx$

d) $\int \tan^8 x\sec^2 x dx$ *e)* $\int \tan^{-3/2} x\sec^4 x dx$ *f)* $\int \tan^4 x\sec^4 x\,dx$

Solution: Since the power of $\sec x$ is even in all of the integrals, use $u = \tan x, du = \sec^2 x dx$ and the identity $\sec^2 x = \tan^2 x + 1$.

a) $\int \tan^2 x\sec^4 x\,dx = \int \tan^2 x\sec^2 x\sec^2 x dx$ (Notice: $du = \sec^2 x dx$)

$$= \int \tan^2 x(\tan^2 x+1)\sec^2 x dx = \int u^2(u^2+1)du$$

$$= \int (u^4+u^2)du = \frac{1}{5}u^5 + \frac{1}{3}u^3 + c = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + c$$

b) $\int \tan^{12} x\sec^4 x dx = \int \tan^{12} x\sec^2 x.\sec^2 x dx$

$$= \int u^{12} du = \frac{1}{13}u^{13} = \frac{1}{13}\tan^{13} x + c$$

c) $\int \tan^4 x\sec^6 x\,dx = \int \tan^4 x(\sec^2 x)^2 \sec^2 x\,dx = \int \tan^4 x(\tan^2 x+1)^2 \sec^2 x\,dx$

$$= \int u^4(u^2+1)^2 du = \int (u^8 + 2u^6 + u^4)du$$

$$= \frac{1}{9}u^9 + \frac{2}{7}u^7 + \frac{1}{5}u^5 + c = \frac{1}{9}\tan^9 x + \frac{2}{7}\tan^7 x + \frac{1}{5}\tan^5 x + c$$

d) $\int \tan^8 x\sec^2 x\,dx = \int u^8 du = \frac{1}{9}u^9 + c = \frac{1}{9}\tan^9 x + c$

e) $\int \tan^{-3/2} x\sec^4 x\,dx = \int \tan^{-3/2} x(\tan^2 x+1)\sec^2 x\,dx = \int (u^{1/2} + u^{-3/2})du$

$$= \frac{2}{3}u^{3/2} - 2u^{-1/2} + c = \frac{2}{3}\tan^{3/2} x - 2\tan^{-1/2} x + c$$

f) $\int \tan^4 x\sec^4 x\,dx = \int \tan^4 x\sec^2 x\sec^2 x\,dx = \int \tan^4 x(\tan^2 x+1)\sec^2 x\,dx$

$$= \int u^4(u^2+1)du = \int (u^6 + u^4)du$$

$$= \frac{1}{7}u^7 + \frac{1}{5}u^5 + c = \frac{1}{7}\tan^7 x + \frac{1}{5}\tan^5 x + c$$

Case-III: If m is even and n is odd, using $\tan^2 x = \sec^2 x - 1$, reduce the integrand into powers of $\sec x$ and apply reduction formula of secant function.

Examples: Evaluate the following integrals.

a) $\int \tan^2 x\sec x\,dx$ b) $\int \tan^4 x\sec^3 x\,dx$ c) $\int \tan^2 x\sec^3 x\,dx$

Solution: Since the power of $\tan x$ is even and that of $\sec x$ is odd, using $\tan^2 x = \sec^2 x - 1$, reduce the integrand into powers of $\sec x$.

a) $\int \tan^2 x\sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx = \int (\sec^3 x - \sec x)dx$

$$= \int \sec^3 x\,dx - \int \sec x\,dx = \frac{\sec x\tan x}{2} + \frac{1}{2}\int \sec x\,dx - \int \sec x\,dx$$

$$= \frac{\sec x\tan x}{2} - \frac{1}{2}\int \sec x\,dx = \frac{\sec x\tan x}{2} - \frac{1}{2}\ln|\sec x + \tan x| + c$$

b) $\int \tan^4 x\sec^3 x\,dx = \int (\sec^2 x - 1)^2 \sec^3 x\,dx = \int (\sec^4 x - 2\sec^2 x + 1)\sec^3 dx$

$$= \int \sec^7 x\,dx - \int 2\sec^5 x\,dx + \int \sec^3 x\,dx$$

(Complete using reduction formula of $\sec x$)

IV) ***Integrals of the form*** $\int \cot^m x \csc^n x dx$:

Case-I: If m is odd, use the substitution $u=\csc x$ and $\csc^2 x=\cot^2 x+1$.

Example: Evaluate the following integrals.

a) $\int \cot^3 x\csc^2 xdx$ b) $\int \cot x\csc^9 xdx$ c) $\int \cot^3 x\csc^5 xdx$

d) $\int \cot^3 x\csc^4 xdx$ e) $\int \cot^3 x\csc^3 xdx$ f) $\int \cot^5 x\csc^7 xdx$

Solution: Since the power of $\cot x$ is odd in all of the integrals, use $u=\csc x, du=-\csc x\cot xdx$ and the identity $\csc^2 x=\cot^2 x+1$.

a) $\int \cot^3 x\csc^2 xdx=\int(\csc^2 x-1)\csc x(\csc x\cot x)dx=\int(u-u^3)du$

$$=\frac{u^2}{2}-\frac{1}{4}u^4+c=\frac{\csc^2 x}{2}-\frac{1}{4}\csc^4 x+c$$

b) $\int \cot x\csc^9 xdx=\int \csc^8 x\csc x\cot xdx=-\int u^8du=\frac{-u^9}{9}+c=\frac{-\csc^9 x}{9}+c$

c) $\int \cot^3 x\csc^5 xdx=\int(\csc^2 x-1)\csc^4 x\csc x\cot xdx$

$$=-\int(u^2-1)u^4du=\int(u^4-u^6)du$$

$$=\frac{1}{5}u^5-\frac{1}{7}u^7+c=\frac{1}{5}\csc^5 x-\frac{1}{7}\csc^7 x+c$$

d) $\int \cot^3 x\csc^4 xdx=\int \cot^2 x\csc^3 x\csc x\cot xdx$

$$=\int(\csc^2 x-1)\csc^3 x\csc x\cot xdx$$

$$=-\int(u^2-1)u^3du=\int(u^3-u^5)du$$

$$=\frac{1}{4}u^4-\frac{1}{6}u^6+c=\frac{1}{4}\csc^4 x-\frac{1}{6}\csc^6 x+c$$

e) $\int \cot^3 x\csc^3 xdx=\int(\csc^2 x-1)\csc^2 x\csc x\cot xdx$

$$=-\int(u^2-1)u^2du=\frac{1}{3}u^3-\frac{1}{5}u^5+c$$

$$=\frac{1}{3}\csc^3 x-\frac{1}{5}\csc^5 x+c$$

f) $\int \cot^5 x \csc^7 x dx = \int \cot^4 x \csc^6 x \csc x \cot x dx$

$$= \int (\cot^2 x)^2 \csc^6 x \csc x \cot x dx$$

$$= \int (\csc^2 x - 1)^2 \csc^6 x \csc x \cot x dx = -\int (u^2 - 1)^2 u^6 du$$

$$= -\int (u^4 - 2u^2 + 1) u^6 du = \int (-u^{10} + 2u^8 - u^6) du$$

$$= \frac{-u^{11}}{11} + \frac{2u^9}{9} - \frac{u^7}{7} + c = \frac{-\csc^{11} x}{11} + \frac{2\csc^9 x}{9} - \frac{\csc^7 x}{7} + c$$

Case-II: If n is even, use the substitution $u = \cot x$ and the identity $\csc^2 x = \cot^2 x + 1$.

Example: Evaluate the following integrals.

a) $\int \cot^4 x \csc^4 x dx$ *b)* $\int \cot^{12} x \csc^4 x dx$ *c)* $\int \cot^{19} x \csc^2 x dx$

Solution: Since the power of $\csc x$ is even in all of the integrals, use $u = \cot x, du = -\csc^2 x dx$ and the identity $\csc^2 x = \cot^2 x + 1$.

a) $\int \cot^4 x \csc^4 x dx = \int \cot^4 x \csc^2 x \csc^2 x dx$

$$= \int \cot^4 x (\cot^2 x + 1) \csc^2 x dx$$

$$= -\int u^4 (u^2 + 1) du = -\int (u^6 + u^4) du$$

$$= \frac{-u^7}{7} - \frac{u^5}{5} + c = \frac{-\cot^7 x}{7} - \frac{\cot^5 x}{5} + c$$

b) $\int \cot^{12} x \csc^4 x dx = \int \cot^{12} x \csc^2 x \csc^2 x dx$

$$= \int \cot^{12} x (\cot^2 x + 1) \csc^2 x dx$$

$$= -\int u^{12} (u^2 + 1) du = -\int (u^{14} + u^{12}) du$$

$$= \frac{-u^{15}}{15} - \frac{u^{13}}{13} + c = \frac{-\cot^{15} x}{15} - \frac{\cot^{13} x}{13} + c$$

c) $\int \cot^{19} x \csc^2 x dx = -\int u^{19} du = \frac{-u^{20}}{20} + c = \frac{-\cot^{20} x}{20} + c$

6.2.4 Integration by Trigonometric Substitutions

Trigonometric substitutions is a method used to evaluate integrals that contain expressions of the form $\sqrt{a^2-b^2x^2}$, $\sqrt{b^2x^2+a^2}$ and $\sqrt{b^2x^2-a^2}$ by making substitutions involving trigonometric functions. The basic idea in using trigonometric substitution for an integral that involves one of above forms is to eliminate the radical and change the integrand into a simpler one. Now, let's see one by one how to integrate such forms of integrals.

I) Sine Substitution: For integrals containing $\sqrt{a^2-b^2x^2}$

Use the trigonometric substitution $x=\frac{a}{b}\sin\theta,\ dx=\frac{a}{b}\cos\theta d\theta$. Restricting the domain to be $-\pi/2\le\theta\le\pi/2$, this substitution simplifies the radical as follows:

$$\sqrt{a^2-b^2x^2}=\sqrt{a^2-a^2\sin^2\theta}=a\sqrt{1-\sin^2\theta}=a\cos\theta$$

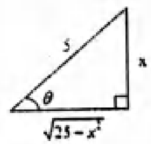
Examples: Evaluate the following integrals

$a)\int\sqrt{25-x^2}dx$ $\quad b)\int\frac{x}{\sqrt{16-x^2}}dx$ $\quad c)\int\sqrt{9-4x^2}dx$ $\quad d)\int\frac{dx}{x^2\sqrt{4-x^2}}$

$e)\int\frac{x^3}{\sqrt{1-x^2}}dx$ $\quad f)\int\frac{x^2}{\sqrt{9-x^2}}dx$ $\quad g)\int\frac{x^2}{(4-x^2)^{\frac{3}{2}}}dx$ $\quad h)\int\frac{1}{(1-x^2)^{\frac{3}{2}}}dx$

Solution:

a) Here, $a=5,\ b=1$. So, let $x=5\sin\theta,\ dx=5\cos\theta d\theta$. Then, from the right angle triangle, $\sin\theta=\frac{x}{5}\Rightarrow\cos\theta=\frac{\sqrt{25-x^2}}{5}$, $\sin2\theta=2\sin\theta\cos\theta$. Therefore,



$$\int\sqrt{25-x^2}dx=5\int\sqrt{25-25\sin^2\theta}\cos\theta d\theta$$

$$=25\int\cos^2\theta d\theta=25\int\frac{1+\cos2\theta}{2}d\theta$$

$$=25\int(\frac{1}{2}+\frac{\cos2\theta}{2})d\theta=\frac{25}{2}\theta+\frac{25}{4}\sin2\theta+c$$

$$=\frac{25}{2}\sin^{-1}\frac{x}{5}+\frac{x}{2}\sqrt{25-x^2}+c$$

b) Here, $a=4, b=1$. So, let $x=4\sin\theta,\ dx=4\cos\theta d\theta$.

Then, from the right angle triangle, $\sin\theta=\frac{x}{4}\Rightarrow\cos\theta=\frac{\sqrt{16-x^2}}{4}$.

Hence, $\int\frac{x}{\sqrt{16-x^2}}dx=\int\frac{4\sin\theta}{\sqrt{16-16\sin^2\theta}}4\cos\theta d\theta$

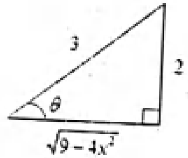
$$=4\int\sin\theta d\theta=-4\cos\theta+c$$

$$=-\sqrt{16-x^2}+c$$

c) Here, $a=3, b=2$. So, let $x=\frac{3}{2}\sin\theta,\ dx=\frac{3}{2}\cos\theta d\theta$.

Besides, from the right angle triangle,

$\sin\theta=\frac{2x}{3}\Rightarrow\cos\theta=\frac{\sqrt{9-4x^2}}{3},\ \sin 2\theta=2\sin\theta\cos\theta=\frac{4x}{9}\sqrt{9-4x^2}$



Hence,

$\int\sqrt{9-4x^2}dx=\int\sqrt{9-9\sin^2\theta}\,\frac{3}{2}\cos\theta d\theta$

$$=\frac{9}{2}\int\cos^2\theta d\theta=\frac{9}{2}\int\frac{1+\cos 2\theta}{2}d\theta$$

$$=\frac{9}{2}\int(\frac{1}{2}+\frac{\cos 2\theta}{2})d\theta=\frac{9}{4}\theta+\frac{9}{8}\sin 2\theta+c$$

$$=\frac{9}{4}\sin^{-1}\frac{2x}{3}+\frac{x}{2}\sqrt{9-4x^2}+c$$

d) Here, $a=2, b=1$. So, let $x=2\sin\theta,\ dx=2\cos\theta d\theta$.

So, $\int\frac{1}{x^2\sqrt{4-x^2}}dx=\int\frac{1}{4\sin^2\theta\sqrt{4-4\sin^2\theta}}2\cos\theta d\theta$

$$=\frac{1}{4}\int\csc^2\theta d\theta$$

$$=-\frac{1}{4}\cot\theta+c$$

$$=-\frac{\sqrt{4-x^2}}{4x}+c$$

e) Here, $a=1$, $b=1$. So, let $x=\sin\theta$, $dx=\cos\theta d\theta$.

Beside, using right angle triangle, $\sin\theta = x \Rightarrow \cos\theta=\sqrt{1-x^2}$.

So, $\int \frac{x^3}{\sqrt{1-x^2}}dx = \int \frac{(\sin\theta)^3\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \int \sin^3\theta d\theta = \int \sin^2\theta \sin\theta d\theta$

$= \int (1-\cos^2\theta)\sin\theta d\theta = \int (1-u^2)du = (\text{using } u=\cos\theta)$

$= u - \frac{1}{3}u^3 + c = \cos\theta - \frac{1}{3}\cos^3\theta + c$

$= \sqrt{1-x^2} - \frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$

f) Let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta d\theta$.

Thus, $\int \frac{x^2}{\sqrt{9-x^2}}dx = \int \frac{(3\sin\theta)^2 3\cos\theta d\theta}{\sqrt{9-9\sin^2\theta}} = 9\int \sin^2\theta d\theta$

$= 9\int (\frac{1-\cos 2\theta}{2})d\theta = \frac{9}{2}\int (1-\cos 2\theta)d\theta$

$= \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta + c = \frac{9}{2}\theta - \frac{9}{2}\sin\theta\cos\theta + c$

$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x\sqrt{9-x^2}}{2} + c$

g) Here, $a=2, b=1$. So, let $x = 2\sin\theta$, $dx = 2\cos\theta d\theta$.

So, $\int \frac{x^2 dx}{(4-x^2)^{\frac{3}{2}}} = \int \frac{4\sin^2\theta(2\cos\theta)d\theta}{(4-4\sin^2\theta)^{\frac{3}{2}}} = \int \frac{8\sin^2\theta}{8\cos^2\theta}d\theta$

$= \int \tan^2\theta d\theta = \int (\sec^2\theta - 1)d\theta = \tan\theta - \theta + c$

$= \frac{x}{\sqrt{4-x^2}} - \sin^{-1}\frac{x}{2} + c$

h) Here, $a=b=1$. So, let $x=\sin\theta$, $dx=\cos\theta d\theta$.

So, $\int \frac{1}{(1-x^2)^{\frac{3}{2}}}dx = \int \frac{\cos\theta d\theta}{(1-\sin^2\theta)^{\frac{3}{2}}} = \int \frac{\cos\theta}{\cos^3\theta}d\theta = \int \frac{1}{\cos^2\theta}d\theta$

$= \int \sec^2\theta d\theta = \tan\theta + c = \frac{x}{\sqrt{1-x^2}} + c$

II) Secant Substitution: For integrals containing $\sqrt{b^2x^2 - a^2}$

Use the trigonometric substitution $x = \frac{a}{b}\sec\theta,\ dx = \frac{a}{b}\sec\theta\tan\theta d\theta$ for

$0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$, the radical terms is removed as follow:

That is $\sqrt{b^2x^2 - a^2} = \sqrt{a^2\sec^2\theta - a^2} = a\sqrt{\sec^2\theta - 1} = a\tan\theta$

Examples: Evaluate the following integrals

a) $\int \frac{1}{x^2\sqrt{4x^2-9}}dx$ b) $\int \frac{\sqrt{x^2-4}}{x^4}dx$ c) $\int \frac{\sqrt{x^2-9}}{x^3}dx$

d) $\int x^2\sqrt{x^2-1}dx$ e) $\int \frac{\sqrt{x^2-49}}{x}dx$ f) $\int \frac{1}{x^3\sqrt{x^2-4}}dx$

g) $\int \frac{1}{x\sqrt{x^2-16}}dx$ h) $\int \frac{1}{x(x^2-4)^{\frac{3}{2}}}dx$ i) $\int \frac{x^2}{\sqrt{x^2-25}}dx$

Solution:

a) Here, $a = 3, b = 2$. So, let $x = \frac{3}{2}\sec\theta$. Then, $dx = \frac{3}{2}\sec\theta\tan\theta d\theta$. Hence,

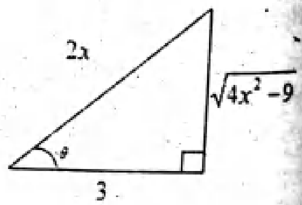
$$\int \frac{1}{x^2\sqrt{4x^2-9}}dx = \int \frac{\frac{3}{2}\sec\theta\tan\theta}{\frac{9}{4}\sec^2\theta\sqrt{9\sec^2\theta - 9}}d\theta$$

$$= \frac{2}{9}\int \frac{1}{\sec\theta}d\theta$$

$$= \frac{2}{9}\int \cos\theta d\theta$$

$$= \frac{2}{9}\sin\theta + c$$

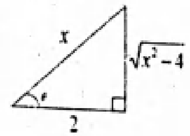
$$= \frac{\sqrt{4x^2-9}}{9x} + c$$



b) Here, $a=2, b=1$. So, let $x=2\sec\theta, dx=\sec\theta\tan\theta d\theta$.

$$\text{So,} \int\frac{\sqrt{x^2-4}}{x^4}dx=\int\frac{\sqrt{4\sec^2\theta-4}}{16\sec^4\theta}.2\sec\theta\tan\theta d\theta$$

$$=\frac{1}{8}\int\frac{\sqrt{4\tan^2\theta}}{\sec^3\theta}.\tan\theta d\theta$$



$$=\frac{1}{8}\int\frac{2\tan^2\theta}{\sec^3\theta}d\theta$$

$$=\frac{1}{4}\int\frac{\sin^2\theta}{\cos^2\theta}.\frac{1}{\sec^3\theta}d\theta$$

$$=\frac{1}{4}\int\frac{\sin^2\theta}{\cos^2\theta}.\cos^3\theta d\theta$$

$$=\tfrac{1}{4}\int\sin^2\theta\cos\theta d\theta$$

Now, use the substitution $u=\sin\theta,\ du=\cos\theta d\theta$, to complete the integral.

That is $\int\frac{\sqrt{x^2-4}}{x^4}dx=\frac{1}{4}\int\sin^2\theta\cos\theta d\theta=\frac{1}{12}\sin^3\theta=\frac{1}{12}\left(\frac{\sqrt{x^2-4}}{x}\right)^3+C$

c) Here, $a=3,\ b=1$. So, let $x=3\sec\theta,\ dx=3\sec\theta\tan\theta d\theta$.

$$\text{Hence,} \int\frac{\sqrt{x^2-9}}{x^3}dx=\int\frac{\sqrt{9\sec^2\theta-9}}{27\sec^3\theta}3\sec\theta\tan\theta d\theta$$

$$=\int\frac{\tan^2\theta}{3\sec^2\theta}d\theta$$

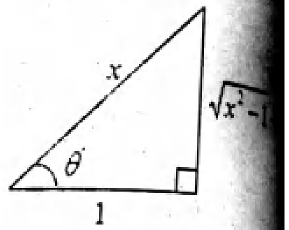
$$=\frac{1}{3}\int\sin^2\theta d\theta=\frac{1}{6}\int(1-\cos2\theta)d\theta$$

$$=\tfrac{1}{6}\theta-\tfrac{1}{12}\sin2\theta+c$$

$$=\tfrac{1}{6}\theta-\tfrac{1}{6}\sin\theta\cos\theta+c$$

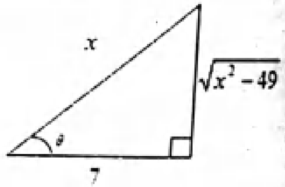
$$=\tfrac{1}{6}\sec^{-1}\tfrac{x}{3}-\frac{\sqrt{x^2-9}}{2x^2}+c$$

d) Here, $a=b=1$. So, let $x=\sec\theta$. Then, $dx=\sec\theta\tan\theta d\theta$. Hence,



$$\int x^2\sqrt{x^2-1}dx=\int\sec^2\theta\sqrt{\sec^2\theta-1}\sec\theta\tan\theta d\theta$$
$$=\int\sec^3\theta\tan^2\theta d\theta$$
$$=\int\sec^3\theta(\sec^2\theta-1)d\theta$$
$$=\int(\sec^5\theta-\sec^3\theta)d\theta$$
$$=\int\sec^5\theta d\theta-\int\sec^3\theta d\theta$$
$$=\frac{x(2x^2-1)\sqrt{x^2-1}}{8}-\frac{1}{8}\ln\left|x+\sqrt{x^2-1}\right|$$

e) Here, $a=7, b=1$. So, let $x=7\sec\theta$, $dx=7\sec\theta\tan\theta d\theta$. Hence,



$$\int\frac{\sqrt{x^2-49}}{x}dx=\int\frac{\sqrt{49\sec^2\theta-49}}{7\sec\theta}7\sec\theta\tan\theta d\theta$$
$$=7\int\tan^2\theta d\theta$$
$$=7\int(\sec^2\theta-1)d\theta$$
$$=7\tan\theta-7\theta+c$$
$$=\sqrt{x^2-49}-7\sec^{-1}\frac{x}{7}+c$$

f) Here, $a=2, b=1$. So, let $x=2\sec\theta$. Then, $dx=2\sec\theta\tan\theta d\theta$. Hence,

$$\int\frac{1}{x^3\sqrt{x^2-4}}dx=\int\frac{2\sec\theta\tan\theta}{8\sec^3\theta\sqrt{4\sec^2\theta-4}}d\theta=\int\frac{1}{8\sec^2\theta}d\theta$$
$$=\frac{1}{8}\int\cos^2\theta d\theta=\frac{1}{8}\int(\frac{1}{2}+\cos2\theta)d\theta$$
$$=\frac{1}{16}\theta+\frac{1}{16}\sin2\theta+c=\frac{1}{16}\theta+\frac{1}{8}\sin\theta\cos\theta+c$$
$$=\frac{1}{16}\sec^{-1}\frac{x}{2}+\frac{\sqrt{x^2-4}}{8x^2}+c$$

g) Here, $a=4, b=1$. So, let $x=4\sec\theta$. Then, $dx=4\sec\theta\tan\theta d\theta$.

Hence, $\int \frac{1}{x\sqrt{x^2-16}}dx=\int \frac{1}{4\sec\theta\sqrt{16\sec^2\theta-16}}4\sec\theta\tan\theta d\theta$

$$=\int \frac{1}{4}d\theta=\frac{1}{4}\theta+c=\frac{1}{4}\sec^{-1}\frac{x}{4}+c$$

h) Here, $a=2, b=1$. So, let $x=2\sec\theta$.

Then, $dx=2\sec\theta\tan\theta d\theta$. Hence,

$$\int \frac{1}{x(x^2-4)^{\frac{3}{2}}}dx=\int \frac{1}{2\sec\theta(4\sec^2\theta-4)^{\frac{3}{2}}}2\sec\theta\tan\theta d\theta$$

$$=\int \frac{\tan\theta}{8\tan^3\theta}d\theta=\frac{1}{8}\int \frac{\cos^2\theta}{\sin^2\theta}d\theta$$

$$=\frac{1}{8}\int \frac{(1-\sin^2\theta)}{\sin^2\theta}d\theta$$

$$=\frac{1}{8}\int \left(\frac{1}{\sin^2\theta}-1\right)d\theta=\frac{1}{8}\int (\csc^2\theta-1)d\theta$$

$$=-\frac{1}{8}\cot\theta-\frac{1}{8}\theta+c$$

$$=-\frac{1}{4\sqrt{x^2-4}}-\frac{1}{8}\sec^{-1}\frac{x}{2}+c$$

i) Here, $a=5, b=1$. So, let $x=5\sec\theta$. Then, $dx=5\sec\theta\tan\theta d\theta$.

Hence, $\int \frac{x^2}{\sqrt{x^2-25}}dx=\int \frac{25\sec^2\theta}{\sqrt{25\sec^2\theta-25}}5\sec\theta\tan\theta d\theta$

$$=25\int \sec^3\theta d\theta$$

$$=\frac{25}{2}\sec\theta\tan\theta+\frac{25}{2}\ln|\sec\theta+\tan\theta|$$

$$=\frac{x\sqrt{x^2-25}}{2}+\frac{25}{2}\ln\left|x+\sqrt{x^2-25}\right|+c$$

III) Tangent Substitution: For integrals containing $\sqrt{b^2x^2+a^2}$.

Use the trigonometric substitution $x = \frac{a}{b}\tan\theta,\ dx = \frac{a}{b}\sec^2\theta d\theta$ for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$\sqrt{b^2x^2+a^2} = \sqrt{b^2(\tfrac{a}{b}\tan\theta)^2+a^2} = \sqrt{a^2\tan^2\theta+a^2} = a\sqrt{\tan^2\theta+1} = a\sec\theta$$

Examples: Evaluate the following integrals

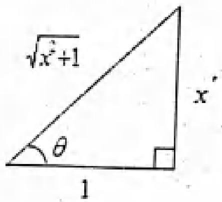
a) $\int \frac{dx}{x^2+1}$ b)* $\int \frac{\sqrt{x^2+4}}{x^4}dx$ c) $\int \frac{x^2}{(x^2+1)^{\frac{5}{2}}}dx$

d) $\int \frac{x^2}{(9+x^2)^2}dx$ e) $\int \frac{dx}{x\sqrt{4+x^2}}$ f)* $\int \frac{1}{x^4\sqrt{x^2+4}}dx$

g) $\int \sqrt{x^2+1}dx$ h) $\int \frac{dx}{x^2\sqrt{x^2+25}}$ i) $\int \frac{x^4}{(x^2+1)^{\frac{7}{2}}}dx$

Solution:

a) Here, $a = b = 1$. So, let $x = \tan\theta$. Then, $dx = \sec^2\theta d\theta$. Hence,



$$\int \frac{1}{x^2+1}dx = \int \frac{1}{\tan^2\theta+1}\sec^2\theta d\theta$$
$$= \int \frac{\sec^2\theta}{\sec^2\theta}d\theta$$
$$= \int d\theta = \theta + c = \tan^{-1}x + c$$

b) Let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta d\theta$.

So, $\int \frac{\sqrt{x^2+4}}{x^4}dx = 2\int \frac{\sqrt{4\tan^2\theta+4}}{16\tan^4\theta}\sec^2\theta d\theta = \frac{1}{4}\int \frac{\sec^3\theta}{\tan^4\theta}d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^4\theta}d\theta$

Now, to evaluate $\int \frac{\cos\theta}{\sin^4\theta}d\theta$, use the substitution $u = \sin\theta \Rightarrow du = \cos\theta d\theta$.

Thus, $\int \frac{\cos\theta}{\sin^4\theta}d\theta = \int \frac{1}{u^4}du = \frac{-1}{3u^3} = \frac{-1}{3\sin^3\theta} = -\frac{1}{3}\csc^3\theta$

But, $x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2} \Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}} \Rightarrow \csc\theta = \frac{\sqrt{x^2+4}}{x}$.

Therefore, $\int \frac{\sqrt{x^2+4}}{x^4}dx = \frac{1}{4}\int \frac{\cos\theta}{\sin^4\theta}d\theta = -\frac{\csc^3\theta}{12}+c$

$$= -\frac{1}{12}\left(\frac{\sqrt{x^2+4}}{x}\right)^3 + c = \frac{-(x^2+4)^{3/2}}{12x^3}+c$$

c) Here, let $x = \tan\theta$. Then, $dx = \sec^2\theta d\theta$. Hence,

$$\int \frac{x^2}{(x^2+1)^{\frac{5}{2}}}dx = \int \frac{\tan^2\theta}{(\tan^2\theta+1)^{\frac{5}{2}}}.\sec^2\theta d\theta = \int \frac{\tan^2\theta}{\sec^3\theta}d\theta$$

$$= \int \frac{\sin^2\theta}{\cos^2\theta}.\cos^3\theta d\theta = \int \sin^2\theta\cos\theta d\theta$$

But $u = \sin\theta, du = \cos\theta d\theta, \int \sin^2\theta\cos\theta d\theta = \int u^2 du = \frac{u^3}{3} = \frac{\sin^3\theta}{3}$

Thus, $\int \frac{x^2}{(x^2+1)^{\frac{5}{2}}}dx = \int \sin^2\theta\cos\theta d\theta = \frac{\sin^3\theta}{3}+c = \frac{1}{3}\left(\frac{x}{\sqrt{x^2+1}}\right)^3 + c$

d) Here, $a = 3$, $b = 1$. So, let $x = 3\tan\theta$. Then, $dx = 3\sec^2\theta d\theta$. Hence,

$$\int \frac{x^2}{(9+x^2)^2}dx = 27\int \frac{(\tan^2\theta)\sec^2\theta d\theta}{(9+9\tan^2\theta)^2}$$

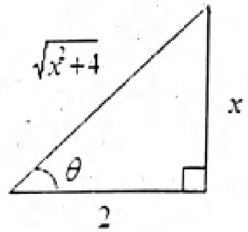
$$= \frac{1}{3}\int \frac{\tan^2\theta d\theta}{\sec^2\theta} = \frac{1}{3}\int \sin^2\theta d\theta$$

$$= \frac{1}{3}\int (\frac{1-\cos 2\theta}{2})d\theta = \frac{1}{6}\theta - \frac{1}{12}\sin 2\theta + c$$

$$= \frac{1}{6}\tan^{-1}\left(\frac{x}{3}\right) - \frac{x}{2(9+x^2)} + c$$

e) Here, $a = 2,\ b = 1$. So, let $x = 2\tan\theta$. Then, $dx = 2\sec^2\theta d\theta$. Hence,

$$\int \frac{1}{x\sqrt{x^2+4}}dx = \int \frac{2\sec^2\theta}{2\tan\theta\sqrt{4\tan^2\theta+4}}d\theta$$



$$= \frac{1}{2}\int \cot\theta\sec\theta d\theta = \frac{1}{2}\int \csc\theta d\theta$$

$$= -\frac{1}{2}\ln\left|\csc\theta + \cot\theta\right| + c$$

$$= -\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}}{x} + \frac{2}{x}\right| + c$$

f) Let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta d\theta$.

Thus, $\displaystyle\int \frac{1}{x^4\sqrt{x^2+4}}dx = \int \frac{1}{16\tan^4\theta\sqrt{4\tan^2\theta+4}}2\sec^2\theta d\theta$

$$= \frac{1}{16}\int \frac{\cos^4\theta\sec\theta}{\sin^4\theta}d\theta = \frac{1}{16}\int \frac{\cos^3\theta}{\sin^4\theta}d\theta$$

Now, to evaluate $\displaystyle\int \frac{\cos^3\theta}{\sin^4\theta}d\theta$. Let's use trigonometric integral technique with $u = \sin\theta \Rightarrow du = \cos\theta d\theta$. Thus,

$$\int \frac{\cos^3\theta}{\sin^4\theta}d\theta = \int \frac{\cos^2\theta\cos\theta}{\sin^4\theta}d\theta = \int \frac{(1-\sin^2\theta)\cos\theta}{\sin^4\theta}d\theta = \int \frac{(1-u^2)}{u^4}du$$

$$= \frac{1}{u} - \frac{1}{3u^3} = \frac{1}{\sin\theta} - \frac{1}{3\sin^3\theta} = \csc\theta - \frac{1}{3}\csc^3\theta$$

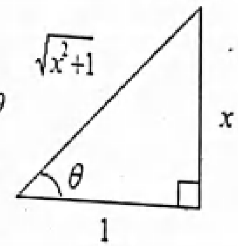
But, $x = 2\tan\theta \Rightarrow \tan\theta = \frac{x}{2} \Rightarrow \sin\theta = \frac{x}{\sqrt{x^2+4}} \Rightarrow \csc\theta = \frac{\sqrt{x^2+4}}{x}$.

Therefore, $\displaystyle\int \frac{1}{x^4\sqrt{x^2+4}}dx = \frac{1}{16}\int \frac{\cos^3\theta}{\sin^4\theta}d\theta = \frac{1}{16}\csc\theta - \frac{1}{48}\csc^3\theta + c$

$$= \frac{\sqrt{x^2+4}}{16x} - \frac{(\sqrt{x^2+4})^3}{48x^3} + c$$

g) Here, $a = b = 1$. So, let $x = \tan\theta$.

Then, $dx = \sec^2\theta d\theta$. Hence,



$$\int\sqrt{x^2+1}dx = \int\sqrt{\tan^2\theta+1}\sec^2\theta d\theta = \int\sec^3\theta d\theta$$

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + c$$

$$= \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|\sqrt{x^2+1}+x\right| + c$$

h) Here, $a = 5,\ b = 1$. So, let $x = 5\tan\theta$. Then, $dx = 5\sec^2\theta d\theta$.

Hence, $\displaystyle\int\frac{dx}{x^2\sqrt{x^2+25}} = \int\frac{5\sec^2\theta}{25\tan^2\theta\sqrt{25\tan^2\theta+25}}d\theta = \frac{1}{25}\int\cot^2\theta\sec\theta d\theta$

$$= \frac{1}{25}\int\csc\theta\cot\theta d\theta = -\frac{1}{25}\csc\theta + c = -\frac{\sqrt{x^2+25}}{25x} + c$$

i) Here, let $x = \tan\theta$. Then, $dx = \sec^2\theta d\theta$. Hence,

$$\int\frac{x^4}{(x^2+1)^{\frac{7}{2}}}dx = \int\frac{\tan^4\theta}{(\tan^2\theta+1)^{\frac{7}{2}}}.\sec^2\theta d\theta = \int\frac{\tan^4\theta}{\sec^7\theta}d\theta$$

$$= \int\frac{\sin^4\theta}{\cos^4\theta}.\cos^7\theta d\theta = \int\sin^4\theta\cos^3\theta d\theta$$

But using $u = \sin\theta, du = \cos\theta d\theta$, we have

$$\int\sin^4\theta\cos^3\theta d\theta = \int\sin^4\theta(1-\sin^2\theta)\cos\theta d\theta$$

$$= \int u^4(1-u^2)du = \int(u^4-u^6)du$$

$$= \frac{u^5}{5} - \frac{u^7}{7} = \frac{\sin^5\theta}{5} - \frac{\sin^7\theta}{7} + c$$

$$= \frac{1}{5}\left(\frac{x}{\sqrt{x^2+1}}\right)^5 - \frac{1}{7}\left(\frac{x}{\sqrt{x^2+1}}\right)^7 + c$$

IV) Integrals involving expressions of the form

ax^2+bx+c **and** $\sqrt{ax^2+bx+c}$:

The above trigonometric substitution methods are also useful to evaluate integrals involving expressions of the form ax^2+bx+c and $\sqrt{ax^2+bx+c}$. This is done based on completing the square method as $(x+a)^2=x^2+2ax+a^2$. That means by ***completing the square***, we can express ax^2+bx+c or

$\sqrt{ax^2+bx+c}$

in terms of $\sqrt{d^2-x^2}, \sqrt{x^2+d^2}$ or $\sqrt{x^2-d^2}$ for properly chosen constant $d>0$. Finally, using trigonometric substitution, we can simplify the integral as earlier.

Examples: Evaluate the following integrals

a) $\int \frac{x\,dx}{\sqrt{3-2x-x^2}}$ b) $\int \frac{8}{x^2+6x+25}dx$ c) $\int \frac{1}{\sqrt{4x^2+4x+17}}dx$

d) $\int \sqrt{3-2x-x^2}\,dx$ e) $\int \sqrt{4x-x^2}\,dx$ f) $\int \frac{x}{(x+4)\sqrt{x^2+8x+25}}dx$

Solution:

a) $\int \frac{x}{\sqrt{3-2x-x^2}}dx=\int \frac{xdx}{\sqrt{4-(1+x)^2}}$.

So using $1+x=2\sin\theta, dx=2\cos\theta d\theta$, we have

$$\int \frac{x}{\sqrt{3-2x-x^2}}dx=\int \frac{x}{\sqrt{4-(1+x)^2}}dx=\int \frac{2\sin\theta-1}{\sqrt{4-4\sin^2\theta}}2\cos\theta d\theta$$

$$=\int (2\sin\theta-1)d\theta=-2\cos\theta-\theta+c$$

$$=-\sqrt{3-2x-x^2}-\sin^{-1}\left(\frac{x+1}{2}\right)+c$$

b) Here, $x^2+6x+25=(x+3)^2+16=u^2+16,\ u=x+3,\ dx=du$

Thus, $\int \frac{8}{x^2+6x+25}dx=\int \frac{8}{(x+3)^2+16}dx=8\int \frac{1}{u^2+16}du$

$$=2\tan^{-1}\frac{u}{4}+c=2\tan^{-1}(\frac{x+3}{4})+c$$

c) Here, $4x^2+4x+17=(2x+1)^2+16=u^2+16,\ \ u=2x+1,\ dx=\frac{du}{2}$

Thus, $\int\frac{1}{\sqrt{4x^2+4x+17}}dx=\int\frac{1}{\sqrt{(2x+1)^2+16}}dx=\frac{1}{2}\int\frac{1}{\sqrt{u^2+16}}du$

$$=\frac{1}{2}\ln\left|\frac{u+\sqrt{u^2+16}}{4}\right|=\frac{1}{2}\ln\left|\frac{2x+1+\sqrt{4x^2+4x+17}}{4}\right|+c$$

d) Using $1+x=2\sin\theta,\ dx=2\cos\theta d\theta$, we have

$$\int\sqrt{3-2x-x^2}\,dx=\int\sqrt{4-(1+x)^2}\,dx=\int\sqrt{4-4\sin^2\theta}\,2\cos\theta d\theta$$

$$=4\int\cos^2\theta d\theta=2\int(1+\cos2\theta)d\theta$$

$$=2\theta+\sin2\theta+c=2\theta+2\sin\theta\cos\theta+c$$

$$=2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+c$$

e) Using $2-x=2\sin\theta,\ dx=-2\cos\theta d\theta$, we have

$$\int\sqrt{4x-x^2}\,dx=\int\sqrt{4-(2-x)^2}\,dx=-2\int\sqrt{4-4\sin^2\theta}\cos\theta d\theta$$

$$=-4\int\cos^2\theta d\theta=-2\int(1+\cos2\theta)d\theta$$

$$=-2\theta-\sin2\theta+c=-2\theta-2\sin\theta\cos\theta+c$$

$$=-2\sin^{-1}\left(\frac{2-x}{2}\right)-\frac{1}{2}(2-x)\sqrt{4x-x^2}+c$$

f) Here, $x^2+8x+25=(x+4)^2+9=u^2+9,\ \ u=x+4,\ dx=du$. Thus,

$$\int\frac{x}{(x+4)\sqrt{x^2+8x+25}}dx=\int\frac{x}{(x+4)\sqrt{(x+4)^2+9}}dx=\int\frac{u-4}{u\sqrt{u^2+9}}du$$

$$=\int\frac{1}{\sqrt{u^2+9}}du-\int\frac{4}{u\sqrt{u^2+9}}du$$

$$=\ln\left|\frac{u+\sqrt{u^2+9}}{3}\right|-\frac{4}{3}\ln\left(\frac{\sqrt{u^2+9}-3}{u}\right)$$

$$=\ln\left|\frac{x+4+\sqrt{x^2+8x+25}}{3}\right|-\frac{4}{3}\ln\left(\frac{\sqrt{x^2+8x+25}-3}{x+4}\right)+c$$

6.2.5 Integration by Partial Fractions

The procedure to integrate rational functions of the form $R(x)=\dfrac{P(x)}{Q(x)}$ using partial fraction is as follow:

Step 1: Factorize or express $Q(x)$ as a product of irreducible factors.

Step 2: Decompose the integrand into sum of rational functions whose denominators involve the factors of $Q(x)$ using arbitrary constants.

The decomposition is done as follow:

i) For every non-repeated linear factor $x-r$, assign a term like $\dfrac{A}{x-r}$.

ii) For twice repeated linear factor like $(x-r)^2$, assign terms $\dfrac{A_1}{x-r}+\dfrac{A_2}{(x-r)^2}$ and for three times repeated linear factor like $(x-r)^3$, assign terms $\dfrac{A_1}{x-r}+\dfrac{A_2}{(x-r)^2}+\dfrac{A_3}{(x-r)^3}$.

iii) For every quadratic factor ax^2+bx+c, assign a term $\dfrac{A_1x+A_2}{ax^2+bx+c}$.

iv) For repeated quadratic factor like $(ax^2+bx+c)^2$, assign terms like

$$\frac{A_1x+A_2}{ax^2+bx+c}+\frac{A_3x+A_4}{(ax^2+bx+c)^2}$$

In each of the above forms A_i's are constants to be determined.

Step 3: Determine the constant A_i's. To determine the constants involved in the decomposition, we can use equality of polynomials or selected values of x

Step 4: Integrate the decomposed integral term by term.

Caution: To apply the above procedures, first be sure that in $R(x)=\dfrac{P(x)}{Q(x)}$, the degree of $P(x)$ is less than the degree of $Q(x)$. If not, divide $P(x)$ by $Q(x)$ and find the remainder $r(x)$ and the quotient $q(x)$. Then, apply the procedures.

Examples:

1. Evaluate the following integrals

$a)\int\frac{1}{x^2-9}dx \quad b)\int\frac{2x-7}{x^2-5x+6}dx \quad c)\int\frac{5x^2+4}{x^3-4x}dx \quad d)\int\frac{5x+9}{x^3-4x^2+3x}dx$

Solution:

a) Here, $Q(x)=x^2-9=(x-3)(x+3)$. Then,

$$\frac{1}{x^2-9}=\frac{1}{(x-3)(x+3)}=\frac{A_1}{x-3}+\frac{A_2}{x+3}.$$

To find the constants, first find the sum of the right hand side and equate coefficients of the same powers of x.

$$\frac{1}{x^2-9}=\frac{1}{(x-3)(x+3)}=\frac{A_1}{x-3}+\frac{A_2}{x+3}=\frac{A_1(x+3)+A_2(x-3)}{(x-3)(x+3)}$$

$$\Rightarrow A_1(x+3)+A_2(x-3)=1\Rightarrow(A_1+A_2)x+3A_1-3A_2=1$$

$$\Rightarrow A_1+A_2=0,3A_1-3A_2=1\Rightarrow A_1=\frac{1}{6},\ A_2=\frac{-1}{6}$$

Hence, $\int\frac{1}{x^2-9}dx=\int\frac{1/6}{x-3}dx-\int\frac{1/6}{x+3}dx$

$$=\frac{1}{6}\ln|x-3|-\frac{1}{6}\ln|x+3|+c=\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right|+c$$

b) Here, $Q(x)=x^2-5x+6=(x-2)(x-3)$. Then,

$\frac{2x-7}{x^2-5x+6}=\frac{2x-7}{(x-2)(x-3)}=\frac{A_1}{x-2}+\frac{A_2}{x-3}$. To find the constants, first find the sum of the right hand side and equate coefficients of the same powers of x.

$$\frac{2x-7}{x^2-5x+6}=\frac{2x-7}{(x-2)(x-3)}=\frac{A_1}{x-2}+\frac{A_2}{x-3}=\frac{A_1(x-3)+A_2(x-2)}{(x-2)(x-3)}$$

$$\Rightarrow A_1(x-3)+A_2(x-2)=2x-6$$

$$\Rightarrow (A_1+A_2)x-3A_1-2A_2=2x-7$$

$$\Rightarrow A_1+A_2=2,-3A_1-2A_2=-7\Rightarrow A_1=3,\ A_2=-1$$

Hence, $\int\frac{2x-7}{x^2-5x+6}dx=\int\frac{3}{x-2}dx-\int\frac{1}{x-3}dx=3\ln|x-2|-\ln|x-3|+c$

c) Here, $Q(x) = x^3 - 4x = x(x-2)(x+2)$. Then,

$$\frac{5x^2+4}{x^3-4x} = \frac{5x^2+4}{x(x-2)(x+2)} = \frac{A_1}{x} + \frac{A_2}{x-2} + \frac{A_3}{x+2}$$

$$\Rightarrow \frac{5x^2+4}{x^3-4x} = \frac{A_1(x-2)(x+2) + A_2x(x+2) + A_3x(x-2)}{x(x-2)(x+2)}$$

$$\Rightarrow A_1(x-2)(x+2) + A_2x(x+2) + A_3x(x-2) = 5x^2+4$$

$$\Rightarrow (A_1 + A_2 + A_3)x^2 + (2A_2 - 2A_3)x - 4A_1 = 5x^2 + 4$$

$$\Rightarrow -4A_1 = 4, A_1 + A_2 + A_3 = 5, 2A_2 - 2A_3 = 0$$

$$\Rightarrow A_1 = -1, A_1 + A_2 + A_3 = 5, A_2 = A_3$$

$$\Rightarrow A_1 = -1, 2A_2 = 6 \Rightarrow A_1 = -1, A_2 = A_3 = 3$$

Hence, $\int \frac{5x^2+4}{x^3-4x}dx = \int \frac{A_1}{x}dx + \int \frac{A_2}{x-2}dx + \int \frac{A_3}{x+2}dx$

$$= \int \frac{-1}{x}dx + \int \frac{3}{x-2}dx + \int \frac{3}{x+2}dx$$

$$= -\ln|x| + 3\ln|x-2| + 3\ln|x+2| = \ln\left|\frac{(x^2-4)^3}{x}\right| + c$$

2. Evaluate the following integrals

a) $\int \frac{18}{x^3-6x^2+9x}dx$ b) $\int \frac{x^3-4x-1}{x(x-1)^3}dx$ c) $\int \frac{12x}{(x-1)(x-2)^3}dx$

Solution:

a) Here, $Q(x) = x^3 - 6x^2 + 9x = x(x-3)^2$. We have repeated linear factor.

$$\frac{18}{x^3-6x^2+9x} = \frac{18}{x(x-3)(x-3)} = \frac{A_1}{x} + \frac{B_1}{x-3} + \frac{B_2}{(x-3)^2}$$

$$= \frac{A_1(x-3)(x-3) + B_1x(x-3) + B_2x}{x(x-3)(x-3)}$$

$$\Rightarrow A_1(x-3)(x-3) + B_1x(x-3) + B_2x = 18$$

$$\Rightarrow (A_1 + B_1)x^2 + (-6A_1 - 3B_1 + B_2)x + 9A_1 = 18$$

$$\Rightarrow A_1 + B_1 = 0, -6A_1 - 3B_1 + B_2 = 0, 9A_1 = 18$$

$$\Rightarrow A_1 = 2, B_1 = -2, B_2 = 6$$

Hence, $\int \frac{18}{x^3-6x^2+9x}dx = \int \frac{A_1}{x}dx + \int \frac{B_1}{x-3}dx + \int \frac{B_2}{(x-3)^2}dx$

$$= \int \frac{2}{x}dx + \int \frac{-2}{x-3}dx + \int \frac{6}{(x-3)^2}dx$$

$$= 2\ln|x| - 2\ln|x-3| - \frac{6}{x-3} + c$$

b) Here, $Q(x) = x(x-1)^3$. We have repeated linear factor.

So, the decomposition is done as follow:

$$\frac{x^3-4x-1}{x(x-1)^3} = \frac{A_1}{x} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{B_3}{(x-1)^3}$$

$$= \frac{A_1(x-1)^3 + B_1x(x-1)^2 + B_2x(x-1) + B_3x}{x(x-1)^3}$$

$$\Rightarrow A_1(x-1)^3 + B_1x(x-1)^2 + B_2x(x-1) + B_3x = x^3 - 4x - 1$$

$$\Rightarrow (A_1+B_1)x^3 + (-3A_1 - 2B_1 + B_2)x^2 + (3A_1 + B_1 - B_2 + B_3)x - A_1 = x^3 - 4x - 1$$

$$\Rightarrow A_1 + B_1 = 1,\ -3A_1 - 2B_1 + B_2 = 0,\ 3A_1 + B_1 - B_2 + B_3 = -4,\ -A_1 = -1$$

$$\Rightarrow A_1 = 1,\ B_1 = 0,\ B_2 = 3,\ B_3 = -4$$

Hence, $\int \frac{x^3-4x-1}{x(x-1)^3}dx = \int \frac{A_1}{x}dx + \int \frac{B_1}{x-1}dx + \int \frac{B_2}{(x-1)^2}dx + \int \frac{B_3}{(x-1)^3}dx$

$$= \int \frac{1}{x}dx + \int \frac{3}{(x-1)^2}dx + \int \frac{-4}{(x-1)^3}dx$$

$$= \ln|x| - \frac{3}{(x-1)} + \frac{2}{(x-1)^2} + c$$

3. Evaluate the following integrals.

$a)\int \frac{x^2+27}{x^3+9x}dx$ $\quad b)\int \frac{x^2+6}{(x-2)(x^2+1)}dx$ $\quad c)\int \frac{3x^2+7}{x^3-x^2+x-1}dx$

$d)\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx$ $\quad e)\int \frac{x^3+5}{(x^2+1)(x^2+2)}dx$ $\quad f)\int \frac{x+4}{x^3+4x}dx$

$g)\int \frac{10}{x(x^2+5)}dx$ $\quad h)\int \frac{x^2+8}{(x^2+1)^2}dx$

Solution:

a) Here, we have mixed factors $x^3+9x = x(x^2+9)$.

Then, by assign appropriate decomposition for both factors:

That is $\frac{x^2+27}{x^3+9x} = \frac{x^2+27}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x(x^2+9)}$

$$\Rightarrow \frac{x^2+27}{x^3+9x} = \frac{x^2+27}{x(x^2+9)} = \frac{A(x^2+9)+Bx^2+Cx}{x(x^2+9)}$$

$$\Rightarrow x^2+27 = (A+B)x^2 + Cx + 9A$$

$$\Rightarrow 9A = 27, A+B = 1, C = 0 \Rightarrow A = 3, B = -2, C = 0$$

Therefore, $\int \frac{x^2+27}{x^3+9x}dx = \int \frac{3}{x}dx - \int \frac{2x}{x^2+9}dx = 3\ln|x| - \ln(x^2+9) + C$.

b) Here, $Q(x) = (x-2)(x^2+1)$. Then,

$$\frac{x^2+6}{(x-2)(x^2+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+Bx(x-2)+C(x-2)}{x^3-2x^2+x-2}$$

$$\Rightarrow (A+B)x^2 + (-2B+C)x + A - 2C = x^2+6$$

$$\Rightarrow A = 2, \ B = -1, \ C = -2$$

Hence, $\int \frac{x^2+6}{(x-2)(x^2+1)}dx = \int \frac{A}{x-2}dx + \int \frac{Bx+C}{x^2+1}dx = \int \frac{2}{x-2}dx + \int \frac{-x-2}{x^2+1}dx$

$$= 2\ln|x-2| - \tfrac{1}{2}\ln(x^2+1) - 2\tan^{-1}x + c$$

c) Here, watch carefully: $x^3 - x^2 + x - 1 = (x-1)(x^2+1)$.

Then, by assign appropriate decomposition for both factors:

$$\frac{3x^2+7}{x^3-x^2+x-1} = \frac{3x^2+7}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$\Rightarrow \frac{3x^2+7}{(x-1)(x^2+1)} = \frac{A(x^2+1)+Bx(x-1)+C(x-1)}{(x-1)x(x^2+1)}$$

$$\Rightarrow (A+B)x^2 + (C-B)x + A - C = 3x^2+7$$

$$\Rightarrow A+B = 3, C-B = 0, A-C = 7 \Rightarrow A = 5, \ B = -2, C = -2$$

Hence, $\int \frac{3x^2+7}{x^3-x^2+x-1}dx = \int \frac{5}{x-1}dx + \int \frac{-2x-2}{x^2+1}dx$

$$= 5\int \frac{1}{x-1}dx - \int \frac{2x}{x^2+1}dx - \int \frac{2}{x^2+1}dx$$

$$= 5\ln|x-1| - \ln(x^2+1) + 2\tan^{-1}x + c$$

d) $\frac{2x^3-4x-8}{(x^2-x)(x^2+4)} = \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+4} \Rightarrow A=2, B=-2, C=2, D=4$

So, $\int \frac{2x^3-4x-8}{(x^2-x)(x^2+4)}dx = 2\ln|x| - 2\ln|x-1| + \ln(x^2+4) + 2\tan^{-1}\frac{x}{2} + c$

e) $\frac{x^3+5}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$

$$= \frac{Ax(x^2+2)+B(x^2+2)+Cx(x^2+1)+D(x^2+1)}{x^4+3x^2+2}$$

$$\Rightarrow (A+C)x^3 + (B+D)x^2 + (2A+C)x + 2B + D = x^3 + 5$$

$$\Rightarrow A+C=1,\ B+D=0,\ 2A+C=0,\ 2B+D=5$$

$$\Rightarrow A=-1,\ B=5,\ C=2,\ D=-5$$

Hence, $\frac{x^3+5}{(x^2+1)(x^2+2)} = \int \frac{-x+5}{x^2+1}dx + \int \frac{2x-5}{x^2+2}dx$

$$= -\int \frac{x}{x^2+1}dx + 5\int \frac{1}{x^2+1}dx + \int \frac{2x}{x^2+2}dx - 5\int \frac{1}{x^2+2}dx$$

$$= -\tfrac{1}{2}\ln(x^2+1) + 5\tan^{-1}x + \ln(x^2+2) - \tfrac{5}{\sqrt{2}}\tan^{-1}\tfrac{x}{\sqrt{2}}$$

f) Here, $\frac{x+4}{x^3+4x} = \frac{x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4} = \frac{A(x^2+4)+Bx^2+Cx}{x(x^2+4)}$

$$\Rightarrow A(x^2+4) + Bx^2 + Cx = x+4$$

$$\Rightarrow 4A=4, A+B=0, C=1 \Rightarrow A=1,\ B=-1$$

Hence, $\int \frac{x+4}{x^3+4x}\,dx = \int \frac{1}{x}\,dx + \int \frac{-x+1}{x^2+4}\,dx$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+4) + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

g) Here, we have mixed factors.

So, the partial fraction decomposition is of the form

$$\frac{10}{x(x^2+5)} = \frac{A}{x} + \frac{Bx+C}{x^2+5} \Rightarrow \frac{10}{x(x^2+5)} = \frac{A(x^2+5)+Bx^2+Cx}{x(x^2+5)}$$

$$\Rightarrow A = 2, B = -2, C = 0$$

Hence, $\int \frac{10}{x(x^2+5)}\,dx = \int \frac{2}{x}dx - \int \frac{2x}{x^2+5}\,dx = 2\ln|x| - \ln(x^2+5) + c.$

6.3 Definite Integral and its Properties

(To understand the more detail concepts how the definition of definite integrals are developed please read about Upper and Lower Riemann Sums.
Here I will only put the final result of the concepts!)

Definition: Let f be continuous on the bounded interval $[a,b]$.Then, the numerical value $\int_a^b f(x)dx$ is called the definite integral of f from a to b where a is called lower limit of integration and b is called upper limit of integration.

We have seen that the indefinite integral of a function results in a family of function. But the definite integral of a function over a given interval is a number.

Properties of Definite Integrals: Let f and g be any two integrable functions on $[a,b]$ and k be any constant. Then,

i) $\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$ ii $\int_b^a f(x)dx = -\int_a^b f(x)dx$

iii) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \forall c \in [a,b]$ iv) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

6.3.1 The Fundamental Theorems of Calculus

This fundamental theorem establishes two basic relations about definite and indefinite integrals. The first part of the theorem provides us a means how to evaluate definite integrals from any known indefinite integral of a function and the second part gives us a simple way of finding the derivative of an integral whose limit of integration involves a variable. These two results together are known as *Fundamental Theorems of Calculus*.

The First Fundamental Theorem of Calculus: Suppose f is continuous on an $[a,b]$.If F is any anti-derivative (indefinite integral) of f on $[a,b]$, then

$\int_a^b f(x)dx = F(b) - F(a)$. (The constant of integration is immaterial).

Notation: $F(b) - F(a) = F(x)]_a^b$.

Examples:

1. Evaluate the following definite integrals.

$$a)\int_0^1 4x^2(x^3+1)^3dx \qquad b)\int_{\ln 2}^{\ln 3} e^{3x}dx \qquad c)\int_1^4\left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)dx$$

$$d)\int_0^{\pi/2}\sin^5 x\cos x dx \qquad e)\int_0^{\frac{\pi}{4}}\frac{\sin^3 x}{\cos^2 x}dx$$

Solution:

a) First, find the anti-derivative of the function.

Use the substitution $u=x^3+1 \Rightarrow du=3x^2dx$.

Then, $\int 4x^2(x^3+1)^3dx=\frac{4}{3}\int u^3du=\frac{4}{3}.\frac{u^4}{4}=\frac{1}{3}(x^3+1)^4$.

Now, use the Fundamental Theorem of Calculus:

Hence, $\int_0^1 4x^2(x^3+1)^3dx=\frac{1}{3}(x^3+1)^4\Big|_{x=0}^{x=1}=\frac{1}{3}(16-1)=\frac{1}{3}(15)=5$.

$$b)\int_{\ln 2}^{\ln 3}e^{3x}dx=\frac{e^{3x}}{3}\Big]_{\ln 2}^{\ln 3}=\frac{e^{3\ln 3}}{3}-\frac{e^{3\ln 2}}{3}=\frac{19}{3}$$

$$c)\int_1^4\left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)dx=\int_1^4\sqrt{x}dx+\int_1^4\frac{1}{2\sqrt{x}}dx=\frac{2x^{3/2}}{3}\Big|_1^4+\sqrt{x}\Big|_1^4=\frac{17}{3}$$

$d)$ Let $u=\sin x$, $du=\cos x dx \Rightarrow \int\sin^5 x\cos x dx=\int u^5du=\frac{u^6}{6}=\frac{(\sin x)^6}{6}$.

Thus, $\int_0^{\pi/2}\sin^5 x\cos x dx=\frac{(\sin x)^6}{6}\Big|_0^{\pi/2}=\frac{1}{6}$

2. Evaluate the following definite integrals using properties of definite integral

$$a)\int_{-3}^{6}\left|9-x^2\right|dx \qquad b)\int_0^4 f(x)dx \text{ where } f(x)=\begin{cases}4-x^2; & 0\le x\le 3\\ 3x-14; & 3\le x\le 4\end{cases}$$

Solution:

a) From the property of absolute value, $\left|9-x^2\right|=\begin{cases}9-x^2; & -3\le x\le 3\\ x^2-9; & 3\le x\le 6\end{cases}$

$$\int_{-3}^{6}|9-x^2|dx = \int_{-3}^{3}|9-x^2|dx + \int_{3}^{6}|9-x^2|dx = \int_{-3}^{3}(9-x^2)dx + \int_{3}^{6}(x^2-9)dx$$

$$= 9x - \frac{x^3}{3}\Big]_{-3}^{3} + \frac{x^3}{3} - 9x\Big]_{3}^{6} = 36 + 36 = 72$$

b) $\int_0^4 f(x)dx = \int_0^3 f(x)dx + \int_3^4 f(x)dx = \int_0^3 (4-x^2)dx + \int_3^4 (3x-14)dx = -\frac{1}{2}$

3. Evaluate the integral $\int_1^{e^\pi} \sin(\ln x)dx$ using appropriate technique.

Solution: First, use the substitution $u = \ln x, x = e^u$ such that $dx = e^u du$.

Then, $\int \sin(\ln x)dx = \int e^u \sin u du$. Now, use integration by parts repeatedly.

By LIATE rule, $f(u) = \sin u,\ g'(u) = e^u$ so that $f'(u) = \cos u,\ g(u) = e^u$.

So, $\int e^u \sin u du = e^u \sin u - \int e^u \cos u du = e^u \sin u - (e^u \cos u + \int e^u \sin u du)$

$$= e^u \sin u - e^u \cos u - \int e^u \sin u du$$

$$\Rightarrow 2\int e^u \sin u du = e^u \sin u - e^u \cos u$$

$$\Rightarrow \int e^u \sin u du = \tfrac{e^u}{2}(\sin u - \cos u)$$

Hence, form $u = \ln x$, $\int \sin(\ln x)dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)]$.

Thus, $\int_1^{e^\pi} \sin(\ln x)dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)]\Big|_{x=1}^{x=e^\pi}$

$$= \tfrac{e^\pi}{2}[\sin(\ln e^\pi) - \cos(\ln e^\pi)] - \tfrac{1}{2}[\sin(\ln 1) - \cos(\ln 1)]$$

$$= \tfrac{e^\pi}{2}[\sin(\pi) - \cos(\pi)] - \tfrac{1}{2}[\sin(0) - \cos(0)] = \tfrac{1}{2}(e^\pi + 1)$$

$$= \tfrac{e^\pi}{2}[0+1] - \tfrac{1}{2}[0-1] = \tfrac{1}{2}(e^\pi + 1)$$

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The Second Fundamental Theorems of Calculus:

If f is continuous on an interval I, then it has anti-derivative (indefinite integral) in I. Furthermore, if a is any point in I, then $F(x)=\int_a^x f(t)dt$ is an indefinite integral of f and $F'(x)=\frac{d}{dx}\left(\int_a^x f(t)dt\right)=f(x)$.

Examples: $a)$ $F(x)=\int_\pi^x \sin(3t^3)dt \Rightarrow F'(x)=\sin(3x^3)$

$b)$ $F(x)=\int_0^x \sqrt{1+t^4}\,dt \Rightarrow f'(x)=\sqrt{1+x^4}$

Corollary: I) If $F(x)=\int_a^{g(x)} f(t)dt$, then $F'(x)=f(g(x))g'(x)$.

II) If $F(x)=\int_{h(x)}^{g(x)} f(t)dt$, then $F'(x)=f(g(x))g'(x)-f(h(x))h'(x)$

Examples:

$a)$ $F(x)=\int_0^{x^2} \sqrt{1+t^4}\,dt \Rightarrow F'(x)=\sqrt{1+(x^2)^4}\,(x^2)'=2x\sqrt{1+x^8}$

$b)$ $F(x)=\int_0^{x^3} \sin^6(t^2)dt \Rightarrow F'(x)=x^2\sin^6(x^3)^2(x^3)'=3x^2\sin^6(x^6)$

$c)$ $F(x)=\int_{-3x}^{3x} e^{t^2}dt \Rightarrow F'(x)=e^{(3x)^2}(3x)'-e^{(-3x)^2}(-3x)'=6e^{9x^2}$

2. Find $F'(x)$ where $a)$ $F(x)=\int_{x^3}^{x^2}\cos(t^2+2t)dt$ $\quad b)$ $F(x)=\int_0^{x^2}\sqrt{1+5t^3}\,dt$

Solution: $a)$ $F'(x)=2x\cos(x^4+2x^2)-3x^2\cos(x^6+2x^3)$

$b)$ $F'(x)=\sqrt{1+5(x^2)^3}\,(x^2)'=2x\sqrt{1+5x^6}$

3*. Given $F(x)=\int_1^{x^3}\sqrt{8t^4+1}\,dt$. Then, find $(F^{-1})'(0)$ and write the equation of the tangent line at $x_0=F^{-1}(0)$ to the graph of F.

Solution: First, find a with the property that $F(a)=0$.

That is $F(a)=0 \Rightarrow F(a)=\int_1^{a^3}\sqrt{8t^4+1}\,dt=0 \Rightarrow a^3=1 \Rightarrow a=1$.

Besides, by SFTC, $F'(x)=3x^2\sqrt{8x^{12}+1} \Rightarrow F'(1)=3\sqrt{8+1}=3\sqrt{9}=9$

Therefore, $(F^{-1})'(1)=\frac{1}{F'(1)}=\frac{1}{9}$.

Equation of tangent line: To write the equation, first find the slope. We know that the slope of a tangent line to the graph of f at a given point $x=x_0$ is determined using slope: $m=f'(x_0)$.

But we found that $x_0=F^{-1}(0) \Rightarrow F(x_0)=0 \Rightarrow x_0=1$.

Hence, the slope is $F'(x)=3x^2\sqrt{8x^{12}+1} \Rightarrow m=F'(1)=3\sqrt{9}=9$.

Therefore, using slope intercept form, equation of the tangent line becomes $y-F(x_0)=m(x-x_0) \Rightarrow y-F(1)=9(x-1) \Rightarrow y=9(x-1)$.

6.3.2 Improper Integrals and Their Convergences

The definite integral $\int_a^b f(x)dx$ is called an *improper integral* if and only if

I) The integrand $f(x)$ has one or more points of discontinuity on the interval $[a,b]$

Or

II) At least one of the limit of integration a or b is *infinity*.

Evaluation Techniques of Improper Integrals:

Case-I: When the limit of integration is infinity

a) *Improper integrals of the form* $\int_a^{\infty} f(x)dx$: Suppose $f(x)$ is continuous on $a \le x \le u$. Then, $\int_a^{\infty} f(x)dx = \lim_{u\to+\infty}\int_a^u f(x)dx$.

b) *Improper integrals of the form* $\int_{-\infty}^{b} f(x)dx$: Suppose $f(x)$ is continuous on $l \le x \le b$. Then, $\int_{-\infty}^{b} f(x)dx = \lim_{l\to-\infty}\int_l^b f(x)dx$.

c) *Improper integrals of the form* $\int_{-\infty}^{+\infty} f(x)dx$: Suppose $f(x)$ is continuous on $l \le x \le u$. Then, $\int_{-\infty}^{+\infty} f(x)dx = \lim_{l\to-\infty}\int_l^0 f(x)dx + \lim_{u\to+\infty}\int_0^u f(x)dx$.

Case-II: When the function has points of discontinuity on $[a,b]$:

a) Suppose $f(x)$ is discontinuous at $x=b$. Then, the improper integral $\int_a^b f(x)dx$ is evaluated by $\int_a^b f(x)dx = \lim_{x\to b^-}\int_a^x f(x)dx$.

b) Suppose $f(x)$ is discontinuous at $x=a$. Then, the improper integral $\int_a^b f(x)dx$ is evaluated by $\int_a^b f(x)dx = \lim_{x\to a^+}\int_x^b f(x)dx$.

c) Suppose $f(x)$ is discontinuous at $x=c$ where $a<c<b$ Then, the improper integral $\int_a^b f(x)dx$ is evaluated by $\int_a^b f(x)dx = \lim_{y\to c^-}\int_a^y f(x)dx + \lim_{y\to c^+}\int_y^b f(x)dx$

Test for The Convergence of Improper Integrals:

An improper integral $\int_a^b f(x)dx$ of any kind is said to be convergent if the value of the definite integral is finite and it is said to be divergent if it is infinity.

That is $\int_a^b f(x)dx = \begin{cases} k, -\infty < k < \infty, \text{ it converges} \\ -\infty \text{ or } +\infty, \text{ it diverges} \end{cases}$

Examples:

1. Evaluate each of the following improper integrals and determine whether it converges or diverges.

a) $\int_0^{+\infty} \frac{1}{(x+2)^4}dx$ b) $\int_{-\infty}^0 e^{5x}dx$ c) $\int_0^{+\infty} \frac{2x}{x^2+1}dx$ d) $\int_e^{+\infty} \frac{dx}{x(\ln x)^{3/2}}dx$

Solution: Using the different situations of case-I, we have,

a) $\int_0^{+\infty} \frac{1}{(x+2)^4}dx = \lim_{u\to+\infty}\int_0^u \frac{1}{(x+2)^4}dx = \lim_{u\to+\infty}\int_0^u (x+2)^{-4}dx$

$$= -\frac{1}{3}\lim_{u\to+\infty}\left(\frac{1}{(x+2)^3}\right)\Big|_0^u = -\frac{1}{3}\lim_{u\to+\infty}[\frac{1}{(u+2)^3} - \frac{1}{8}] = \frac{1}{24}$$

Therefore, the improper integral converges.

b) $\int_{-\infty}^0 e^{5x}dx = \lim_{l\to-\infty}\int_l^0 e^{5x}dx = \frac{1}{5}\lim_{l\to-\infty}[1-e^{5l}] = \frac{1}{5}$

c) $\int_0^{+\infty} \frac{2x}{x^2+1}dx = \lim_{u\to\infty}\int_0^u \frac{2x}{x^2+1}dx = \lim_{u\to\infty}\ln(u^2+4)] = \infty$. Therefore, it diverges.

d) $\int_e^{+\infty} \frac{dx}{x(\ln x)^{3/2}}dx = \lim_{u\to\infty}\int_e^u \frac{dx}{x(\ln x)^{3/2}}dx = -2\lim_{u\to\infty}\frac{1}{\sqrt{\ln x}}\Big|_e^u$

$$= -2\lim_{u\to\infty}[\frac{1}{\sqrt{\ln u}} - \frac{1}{\sqrt{\ln e}}] = -2(0-1) = 2$$

2. Evaluate each of the improper integrals and determine the convergences.

$$a)\int_0^4 \frac{1}{\sqrt{x}}dx \quad b)\int_2^6 \frac{1}{x-6}dx \quad c)\int_1^3 \frac{1}{(x-2)^2}dx \quad d)\int_0^4\left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)dx$$

Solution: Using the different situations of case -II, we have,

$$a)\int_0^4 \frac{1}{\sqrt{x}}dx = \lim_{l\to 0^+}\int_l^4 \frac{1}{\sqrt{x}}dx = \lim_{l\to 0^+} 2\sqrt{x}\Big|_l^4 = \lim_{l\to 0^+}[2\sqrt{4}-2\sqrt{l}] = 4-0=4.$$

$$b)\int_2^6 \frac{1}{x-6}dx = \lim_{u\to 6^-}\int_2^u \frac{1}{x-6^-}dx = \lim_{u\to 6^-}\ln|x-6|\Big|_2^u = \lim_{u\to 6^-}[\ln|u-6|-\ln 4] = \infty$$

c) Here, the point of discontinuity is $x=2$

$$\int_1^3 \frac{1}{(x-2)^2}dx = \int_1^2 \frac{1}{(x-2)^2}dx + \int_2^3 \frac{1}{(x-2)^2}dx = \lim_{u\to 2^-}\int_1^u \frac{dx}{(x-2)^2} + \lim_{l\to 2^+}\int_l^3 \frac{dx}{(x-2)^2}$$

$$= \lim_{u\to 2^-}\left(-\frac{1}{x-2}\right)\Bigg|_1^u + \lim_{l\to 2^+}\left(-\frac{1}{x-2}\right)\Bigg|_l^3 = \infty.$$ Therefore, it diverges.

$$d)\int_0^4\left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)dx = \lim_{l\to 0+}\int_l^4\left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)dx = \lim_{l\to 0+}\int_l^4 \sqrt{x}dx + \lim_{l\to 0+}\int_l^4 \frac{1}{2\sqrt{x}}dx$$

$$= \lim_{l\to 0+}\left[\frac{2x^{3/2}}{3}\right]_l^4 + \lim_{l\to 0+}\left[\sqrt{x}\right]_l^4 = \frac{16}{3}+2 = \frac{22}{3}$$

3. Determine whether the improper integrals are convergent or divergent.

$$a)\int_0^{+\infty}\frac{dx}{\sqrt{x}(x+4)} \quad b^*)\int_{-1}^{+\infty}\frac{x^2}{(3+x^6)^{3/2}}dx \quad c)\int_2^{+\infty}\frac{1}{(2x-1)^3}dx \quad d)\int_{-3}^{\infty}(3-x)e^{-x}dx$$

$$e)\int_1^{\infty}\frac{8dx}{x^2+1} \quad f)\int_0^{\infty}\frac{dx}{(x-2)^2} \quad g)\int_2^{\infty}\frac{dx}{x\ln x} \quad h)\int_0^1\frac{dx}{(1-x)^{1/4}}$$

Solution:

a) Here, we use the substitution $u=\sqrt{x} \Rightarrow du = \dfrac{dx}{2\sqrt{x}} \Rightarrow dx = 2udu, x=u^2$. Then,

$$\int_0^{+\infty}\frac{dx}{\sqrt{x}(x+4)} = \int_0^{+\infty}\frac{2du}{u^2+4} = \lim_{l\to 0^+}\int_l^2\frac{2du}{u^2+4} + \lim_{t\to +\infty}\int_2^t\frac{2du}{u^2+4} = \frac{\pi}{2}$$

The improper integral is convergent.

b) Here, let's rearrange for trig substitution. That is $3+x^6=3+(x^3)^2$.

Now let $u=x^3 \Rightarrow du=3x^2dx$. Then, $\int\frac{x^2}{(3+x^6)^{3/2}}dx=\frac{1}{3}\int\frac{1}{(3+u^2)^{3/2}}du$.

Here, by tangent substitution, $u=\sqrt{3}\tan\theta \Rightarrow du=\sqrt{3}\sec^2\theta d\theta$, we have

$$\frac{1}{3}\int\frac{1}{(3+u^2)^{3/2}}du=\frac{1}{3}\int\frac{1}{(3+3\tan^2\theta)^{3/2}}\sqrt{3}\sec^2\theta d\theta=\frac{1}{9}\int\frac{1}{\sec\theta}d\theta$$

$$=\frac{1}{9}\int\cos\theta d\theta=\frac{1}{9}\sin\theta=\frac{u}{9\sqrt{3+u^2}}=\frac{x^3}{9\sqrt{3+x^6}}$$

Hence, $\int_{-1}^{+\infty}\frac{x^2}{(3+x^6)^{3/2}}dx=\lim_{u\to\infty}\int_{-1}^{u}\frac{x^2}{(3+x^6)^{3/2}}dx=\frac{1}{9}\lim_{u\to\infty}\frac{x^3}{\sqrt{3+x^6}}\Bigg|_{-1}^{u}$

$$=\frac{1}{9}\lim_{u\to\infty}[\frac{u^3}{\sqrt{3+u^6}}+\frac{1}{2}]=\frac{1}{9}\lim_{u\to\infty}[1+\frac{1}{2}]=\frac{1}{6}$$

c) $\int_{2}^{+\infty}\frac{1}{(2x-1)^3}dx=\lim_{u\to+\infty}\int_{2}^{u}\frac{1}{(2x-1)^3}dx=-\frac{1}{4}\lim_{u\to\infty}\frac{1}{(2x-1)^2}\Bigg|_{2}^{u}$

$$=-\frac{1}{4}\lim_{u\to\infty}[\frac{1}{(2u-1)^2}-\frac{1}{9}]=-\frac{1}{4}(-\frac{1}{9})=\frac{1}{36}$$

d) $\int_{-3}^{\infty}(3-x)e^{-x}dx=\lim_{u\to+\infty}\int_{-3}^{u}(3-x)e^{-x}dx=5e^3$. The improper integral is convergent.

e) $\int_{1}^{\infty}\frac{8}{x^2+1}dx=\lim_{u\to\infty}\int_{1}^{u}\frac{8}{x^2+1}dx=8\lim_{u\to\infty}\left[\tan^{-1}(u)-\tan^{-1}(1)\right]=2\pi$

Mixed Problems on Techniques of Integrations:

1*. Evaluate the following integrals

$a)\int\frac{\sqrt{16-x^8}}{x^5}dx$ $\quad b)\int\frac{(x^2-1)^{\frac{3}{2}}}{x^6}dx$ $\quad c)\int\frac{x^9}{(1+x^2)^6}dx$ $\quad d)\int x^2\sqrt{x^6+1}dx$

$e)\int\frac{x^3}{\sqrt{2-x^2}}dx$ $\quad f)\int\frac{dx}{(9-16x^2)^{\frac{3}{2}}}$ $\quad g)\int x^3\sqrt{9+4x^2}\,dx$

$h)\int\frac{x^{49}}{1+x^{100}}dx$ $\quad i)\int\frac{x^3}{\sqrt{1-x^8}}dx$ $\quad j)\int\frac{x}{\sqrt{1-x^4}}dx$

Solution:

a) Here, $x^8=(x^4)^2$. Let $u=x^4 \Rightarrow dx=\frac{1}{4x^3}du$. (Using $u=4\sin\theta$)

$$\text{Thus, }\int\frac{\sqrt{16-x^8}}{x^5}dx=\int\frac{\sqrt{16-u^2}}{4x^8}du=\int\frac{\sqrt{16-u^2}}{4u^2}du=\int\frac{\sqrt{16-16\sin^2\theta}}{64\sin^2\theta}.4\cos\theta d\theta$$

$$=\frac{1}{4}\int\frac{\cos^2\theta}{\sin^2\theta}d\theta=\frac{1}{4}\int\frac{1-\sin^2\theta}{\sin^2\theta}d\theta=\frac{1}{4}\int(\frac{1}{\sin^2\theta}-1)d\theta$$

$$=\frac{1}{4}\int(\csc^2\theta-1)d\theta=-\frac{1}{4}\cot\theta-\frac{1}{4}\theta+c$$

$$=-\frac{\sqrt{16-u^2}}{4u}-\frac{1}{4}\sin^{-1}\frac{u}{4}+c=-\frac{\sqrt{16-x^8}}{4x^4}-\frac{1}{4}\sin^{-1}\frac{x^4}{4}+c$$

b) Let $x=\sec\theta, dx=\sec\theta\tan\theta d\theta$

$$\text{Thus, }\int\frac{(x^2-1)^{\frac{3}{2}}}{x^6}dx=\int\frac{(\sec^2\theta-1)^{3/2}}{\sec^6\theta}.\sec\theta\tan\theta d\theta=\int\frac{\tan^3\theta}{\sec^6\theta}.\sec\theta\tan\theta d\theta$$

$$=\int\frac{\tan^4\theta}{\sec^5\theta}d\theta=\int\frac{\sin^4\theta}{\cos^4\theta}.\frac{1}{\sec^5\theta}d\theta, \text{ (Note: }\frac{1}{\sec^5\theta}=\cos^5\theta)$$

$$=\int\frac{\sin^4\theta}{\cos^4\theta}.\cos^5\theta d\theta=\int\sin^4\theta\cos\theta d\theta, \text{ (Use } u=\sin\theta)$$

$$=\frac{u^5}{5}+c=\frac{\sin^5\theta}{5}+c=\frac{1}{5}\left(\frac{\sqrt{x^2-1}}{x}\right)^5+c=\frac{1}{5}\frac{(\sqrt{x^2-1})^5}{x^5}+c$$

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c) Let $x=\tan\theta, dx=\sec^2\theta d\theta$

Thus, $\int\frac{x^9}{(1+x^2)^6}dx=\int\frac{\tan^9\theta}{(1+\tan^2\theta)^6}.\sec^2\theta d\theta=\int\frac{\tan^9\theta}{\sec^{12}\theta}.\sec^2\theta d\theta$

$$=\int\frac{\tan^9\theta}{\sec^{10}\theta}d\theta=\int\frac{\sin^9\theta}{\cos^9\theta}.\frac{1}{\sec^{10}\theta}d\theta, \text{(Note: }\frac{1}{\sec^{10}\theta}=\cos^{10}\theta)$$

$$=\int\frac{\sin^9\theta}{\cos^9\theta}.\cos^{10}\theta d\theta=\frac{1}{10}\frac{x^{10}}{(x^2+1)^5}+c$$

d) Here, $x^6=(x^3)^2$. Let $u=x^3\Rightarrow dx=\frac{1}{3x^2}du$. (Using $u=\tan\theta$)

So, $\int x^2\sqrt{x^6+1}dx=\int x^2\sqrt{(x^3)^2+1}dx=\frac{1}{3}\int\sqrt{u^2+1}du=\frac{1}{3}\int\sqrt{\tan^2\theta+1}\sec^2\theta d\theta$

$$=\frac{1}{3}\int\sec^3\theta d\theta=\frac{\sec\theta\tan\theta}{6}+\frac{1}{6}\ln|\sec\theta+\tan\theta|+c$$

$$=\frac{x^3\sqrt{x^6+1}}{6}+\frac{1}{6}\ln\left|\sqrt{x^6+1}+x^3\right|+c$$

e) Let $x=\sqrt{2}\sin\theta\Rightarrow dx=\sqrt{2}\cos\theta d\theta$.

Thus, $\int\frac{x^3}{\sqrt{2-x^2}}dx2\sqrt{2}\int\sin^3\theta d\theta=2\sqrt{2}\int(1-\cos^2\theta)\sin\theta d\theta$

$$=-2\sqrt{2}\cos\theta+\frac{2\sqrt{2}}{3}\cos^3\theta=-2\sqrt{2-x^2}+\frac{1}{3}(\sqrt{2-x^2})^3$$

f) Let $x=3/4\sin\theta\Rightarrow dx=3/4\cos\theta d\theta$.

Thus, $\int\frac{1}{(9-16x^2)^{\frac{3}{2}}}dx=\frac{1}{36}\int\sec^2\theta d\theta=\frac{1}{36}\tan\theta+c=\frac{x}{9\sqrt{9-4x^2}}+c$

h) $1+x^{100}=1+(x^{50})^2, u=x^{50}, du=50x^{49}$.

Thus, $\int\frac{x^{49}}{1+x^{100}}dx=\frac{1}{50}\int\frac{1}{1+u^2}du=\frac{1}{50}\tan^{-1}u+c=\frac{1}{50}\tan^{-1}x^{50}+c$

i) Let $u=x^4\Rightarrow du=4x^3dx$. Then, $\int\frac{x^3dx}{\sqrt{1-x^8}}=\frac{\sin^{-1}u}{4}+c=\frac{1}{4}\sin^{-1}(x^4)+c$

j) Let $x = 3/2\tan\theta \Rightarrow dx = 3/2\sec^2\theta d\theta$. Thus,

$$\int x^3\sqrt{9+4x^2}\,dx = \int (3/2\tan\theta)^3\sqrt{9+9\tan^2\theta}(3/2)\sec^2\theta d\theta$$

$$= \frac{243}{16}\int(\tan^3\theta\sec^3\theta)d\theta = \frac{243}{16}\int(\tan^2\theta\sec^2\theta)\tan\theta\sec\theta d\theta$$

$$= \frac{243}{16}\int(u^2-1)u^2du = \frac{243}{16}\left(\frac{u^5}{5}-\frac{u^3}{3}\right)+c$$

$$= \frac{243}{16}\left(\frac{\sec^5\theta}{5}-\frac{\sec^3\theta}{3}\right)+c = \frac{(9+4x^2)^{5/2}}{80}-\frac{3(9+4x^2)^{3/2}}{16}+c$$

$h) 1-x^4 = 1-(x^2)^2, u = x^2, du = 2x \Rightarrow \int\frac{x}{\sqrt{1-x^4}}dx = \frac{1}{2}\sin^{-1}u = \frac{1}{2}\sin^{-1}x^2$

2. Evaluate the following integrals using the method of partial fraction.

$a)\int\frac{e^x}{e^{2x}-4}dx$ $\quad b)\int\frac{4e^{2x}+13e^x-9}{e^{2x}+2e^x-3}dx$ $\quad c)\int\frac{\cos x}{\sin^2 x+4\sin x-5}dx$

$d)\int\frac{\sec^2\theta d\theta}{\tan^3\theta-\tan^2\theta}$ $\quad e)\int\frac{\cos\theta d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta+2\sin\theta+3}}$ $\quad f)\int\frac{\sec^2\theta d\theta}{\tan^2\theta+3\tan\theta+2}$

Solution:

a) First, rearrange as $u = e^x \Rightarrow du = e^x dx \Rightarrow \int\frac{e^x}{e^{2x}-4}dx = \int\frac{du}{u^2-4}$.

Here, $\frac{1}{u^2-4} = \frac{1}{(u-2)(u+2)} = \frac{A}{u-2}+\frac{B}{u+2} \Rightarrow A = \frac{1}{4}, B = -\frac{1}{4}$

Therefore, $\int\frac{e^x}{e^{2x}-4}dx = \frac{1}{4}\int\frac{1}{u-2}-\frac{1}{4}\int\frac{1}{u+2} = \frac{1}{4}\ln|e^x-2|-\frac{1}{4}\ln|e^x+2|+c$

b) Here, $u = e^x \Rightarrow du = e^x dx \Rightarrow \int\frac{4e^{2x}+13e^x-9}{e^{2x}+2e^x-3}dx = \int\frac{4u^2+13u-9}{u(u^2+2u-3)}du$

Here, $\frac{4u^2+13u-9}{u(u^2+2u-3)} = \frac{4u^2+13u-9}{u(u-1)(u+3)} = \frac{A}{u}+\frac{B}{u-1}+\frac{C}{u+3}$

$$\Rightarrow A = 3, B = 2, C = -1.$$

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Therefore, using the decomposition, the integral is evaluated as follow:

$$\int\frac{4e^{2x}+13e^{x}-9}{e^{2x}+2e^{x}-3}dx=\int\frac{3du}{u}+\int\frac{2du}{u-1}-\int\frac{du}{u+3}=3x+2\ln\left|e^{x}-1\right|-\ln\left|e^{x}+3\right|$$

c) Here, $u=\sin x\Rightarrow du=\cos x dx$. Thus, $\int\frac{\cos x}{\sin^2 x+4\sin x-5}dx=\int\frac{1}{u^2+4u-5}du$

Now, let's apply themethod of partial fraction.

Here, $\frac{1}{u^2+4u-5}=\frac{1}{(u-1)(u+5)}=\frac{A}{u-1}+\frac{B}{u+5}\Rightarrow A=\frac{1}{6},B=-\frac{1}{6}$

$$\int\frac{1}{u^2+4u-5}du=\frac{1}{6}\int\frac{du}{u-1}-\frac{1}{6}\int\frac{du}{u+5}=\frac{1}{6}\ln\left|u-1\right|-\frac{1}{6}\ln\left|u+5\right|+c.$$

Therefore, $\int\frac{\cos x}{\sin^2 x+4\sin x-5}dx=\frac{1}{6}\ln\left|\sin x-1\right|-\frac{1}{6}\ln\left|\sin x+5\right|+c$

d) Let $u=\tan\theta\Rightarrow du=\sec^2\theta d\theta$. So, $\int\frac{\sec^2\theta d\theta}{\tan^3\theta-\tan^2\theta}=\int\frac{du}{u^3-u^2}$

Now, $\frac{1}{u^3-u^2}=\frac{1}{u^2(u-1)}=\frac{A}{u}+\frac{B}{u^2}+\frac{C}{u-1}\Rightarrow A=B=-1,C=1.$

Thus, $\int\frac{\sec^2\theta d\theta}{\tan^3\theta-\tan^2\theta}=\int\frac{du}{u^3-u^2}=\int\frac{-1}{u}du-\int\frac{1}{u^2}du+\int\frac{1}{u-1}du$

$$=\frac{1}{u}-\ln\left|u\right|+\ln\left|u-1\right|=\frac{1}{\tan\theta}-\ln\left|\tan\theta\right|+\ln\left|\tan\theta-1\right|$$

e) Let $u=\sin\theta\Rightarrow du=\cos\theta d\theta$. Then, (using $t=u+1$)

$$\int\frac{\cos\theta\, d\theta}{(1+\sin\theta)\sqrt{\sin^2\theta+2\sin\theta+3}}=\int\frac{du}{(1+u)\sqrt{(u+1)^2+2}}=\int\frac{dt}{t\sqrt{t^2+2}}$$

Now, by tangent substitution, $t=\sqrt{2}\tan\beta$,

$$\int\frac{dt}{t\sqrt{t^2+2}}=\int\frac{\sqrt{2}\sec^2\beta d\beta}{\tan\beta\sqrt{2\tan^2\beta+2}}=\int\frac{\sqrt{2}\sec^2\beta d\beta}{\tan\beta\sqrt{2}\sec\beta}=\int\frac{\sec\beta d\beta}{\tan\beta}$$

$$=\int\csc\beta d\beta=-\ln\left|\csc\beta+\cot\beta\right|=-\ln\left|\frac{\sqrt{(\sin\theta+1)^2+2}}{\sin\theta+1}+\frac{\sqrt{2}}{\sin\theta+1}\right|$$

3. Integrals that need combination of Methods: Evaluate

$a) \int x\sin^{-1} x dx$ $\quad$ $b) \int x\cos^{-1} x dx$ $\quad$ $c) \int \ln(x^2+4)dx$

Solution:

a) Let $f(x) = \sin^{-1} x, g'(x) = x \Rightarrow f'(x) = \frac{1}{\sqrt{1-x^2}}, g(x) = \frac{x^2}{2}.$

Hence, $\int x\sin^{-1} xdx = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx.$

Here, evaluate $\int \frac{x^2}{\sqrt{1-x^2}}dx$ by using sine substitution, $x = \sin\theta, dx = \cos\theta d\theta$

$$\frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx = \frac{1}{2}\int \frac{\sin^2\theta\cos\theta d\theta}{\sqrt{1-\sin^2\theta}} = \frac{1}{2}\int \sin^2\theta d\theta = \frac{1}{2}\int \frac{1-\cos 2\theta d\theta}{2}$$

$$= \frac{1}{4}\theta - \frac{1}{8}\sin 2\theta + c = \frac{1}{4}\theta - \frac{1}{4}\sin\theta\cos\theta = \frac{1}{4}\sin^{-1} x - \frac{1}{4}x\sqrt{1-x^2}$$

Thus, $\int x\sin^{-1} xdx = \frac{1}{2}x^2 \sin^{-1} x - \frac{1}{4}\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + c$

b) Let $f(x) = \cos^{-1} x, g'(x) = x \Rightarrow f'(x) = \frac{-1}{\sqrt{1-x^2}}, g(x) = \frac{x^2}{2}$

$$\Rightarrow \int x\cos^{-1} xdx = \frac{1}{2}x^2 \cos^{-1} x + \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}dx$$

As we did in part (a), $\int \frac{x^2}{\sqrt{1-x^2}}dx = \frac{1}{4}\sin^{-1} x - \frac{1}{4}x\sqrt{1-x^2}$.

Thus, $\int x\cos^{-1} xdx = \frac{1}{2}x^2 \cos^{-1} x - \frac{1}{4}x\sqrt{1-x^2} + \frac{1}{4}\sin^{-1} x + c$

c) $f(x) = \ln(x^2+4),\ g'(x) = 1 \Rightarrow f'(x) = \frac{2x}{x^2+4},\ g(x) = x.$ Thus,

$$\int \ln(x^2+4)dx = x\ln(x^2+4) - 2\int \frac{x^2}{x^2+4}dx = x\ln(x^2+4) - 2\int (1 - \frac{4}{x^2+4})dx$$

$$= x\ln(x^2+4) + 4\tan^{-1}\frac{x}{2} - 2x + c$$

4. Evaluate the following definite integrals

a) $\int_0^8 \frac{x}{(x+1)^{3/2}}dx$ b) $\int_1^e \frac{\sqrt{\ln x}}{x}dx$ c) $\int_1^2 \frac{1}{\sqrt{4x-x^2}}dx$

Solution:

a) Here, $u = x+1,\ dx = du, x = u-1$. Thus,

$$\int \frac{x}{(x+1)^{3/2}}dx = \int \frac{u-1}{u^{3/2}}du = 2\sqrt{u} + \frac{2}{\sqrt{u}} = 2\sqrt{x+1} + \frac{2}{\sqrt{x+1}}$$

Hence, $\int_0^8 \frac{x}{(x+1)^{3/2}}dx = \left(2\sqrt{x+1} + \frac{2}{\sqrt{x+1}}\right)\Bigg|_0^8 = \frac{20}{3} - 4 = \frac{8}{3}$.

b) Let $u = \ln x$, $du = \frac{1}{x}dx \Rightarrow \int \frac{\sqrt{\ln x}}{x}dx = \int \sqrt{u}du = \frac{2}{3}(\ln x)^{3/2}$.

Thus, $\int_1^e \frac{\sqrt{\ln x}}{x}dx = \frac{2}{3}(\ln x)^{3/2}\Big|_1^e = \frac{2}{3}$

c) Here, $4x - x^2 = 4-(x-2)^2$. So, let $x-2 = 2\sin\theta \Rightarrow dx = 2\cos\theta d\theta$. Thus,

$$\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{dx}{\sqrt{4-(x-2)^2}} = \int \frac{2\cos\theta d\theta}{\sqrt{4-4\sin^2\theta}} = \int d\theta = \theta = \sin^{-1}\left(\frac{x-2}{2}\right)$$

Hence, $\int_1^2 \frac{1}{\sqrt{4x-x^2}}dx = \sin^{-1}\left(\frac{x-2}{2}\right)\Bigg|_1^2 = \frac{\pi}{6}$.

5. Find b such that $\int_0^b \frac{1}{x^2+4}dx = \frac{\pi}{6}$.

Solution: Here, using tangent substitution, $\int \frac{1}{x^2+4}dx = \frac{1}{2}\tan^{-1}\frac{x}{2}$.

Hence, $\int_0^b \frac{1}{x^2+4}dx = \frac{1}{2}\tan^{-1}\frac{b}{2} - \frac{1}{2}\tan^{-1}0 = \frac{\pi}{6} \Rightarrow \frac{1}{2}\tan^{-1}\frac{b}{2} = \frac{\pi}{6} \Rightarrow b = 2\sqrt{3}$

6. Given $F(x) = \int_0^{2x^2} t^4 e^{t^2} dt$. Then, give $F'(x)$.

Solution: By SFTC: $F'(x) = (2x^2)^4 e^{(2x^2)^2}(2x^2)' = 64x^9 e^{4x^4}$

7*. Evaluate the **improper integral** and check whether it converges or diverges.

$a)\ \int_{1}^{\infty}\frac{3}{x(2+\ln x)^4}dx \qquad b)\ \int_{3}^{\infty}\frac{dx}{x^2\sqrt{x^2-9}} \qquad c)\ \int_{0}^{\infty}e^{-3x}dx$

Solution:

a) Using substitution, $t=2+\ln x, dt=\frac{1}{x}dx$

$$\int_{1}^{\infty}\frac{3}{x(2+\ln x)^4}dx=\lim_{u\to\infty}\int_{0}^{u}\frac{3}{x(2+\ln x)^4}dx=\lim_{u\to\infty}\int_{0}^{u}\frac{3}{t^4}dt=\lim_{u\to\infty}\int_{0}^{u}3t^{-4}dt$$

$$=\lim_{u\to\infty}(-\frac{1}{(2+\ln x)^3})\Big|_{x=0}^{x=u}=\lim_{u\to\infty}(-\frac{1}{(2+\ln u)^3}+\frac{1}{8})=\frac{1}{8}$$

b) First, use secant substitution $x=3\sec\theta\tan\theta d\theta, dx=3\sec\theta\tan\theta d\theta$.

$$\int_{3}^{\infty}\frac{1}{x^2\sqrt{x^2-9}}dx=\int_{3}^{c}\frac{dx}{x^2\sqrt{x^2-9}}+\int_{c}^{\infty}\frac{1}{x^2\sqrt{x^2-9}}dx$$

$$=\lim_{l\to3^+}\int_{l}^{c}\frac{dx}{x^2\sqrt{x^2-9}}+\lim_{u\to\infty}\int_{c}^{u}\frac{dx}{x^2\sqrt{x^2-9}}$$

$$=\lim_{l\to3^+}\int_{l}^{c}\frac{3\sec\theta\tan\theta d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}+\lim_{u\to\infty}\int_{c}^{u}\frac{3\sec\theta\tan\theta d\theta}{9\sec^2\theta\sqrt{9\sec^2\theta-9}}$$

$$=\lim_{l\to3^+}\int_{l}^{c}\frac{3\sec\theta\tan\theta d\theta}{27\sec^2\theta\tan\theta}+\lim_{u\to\infty}\int_{c}^{u}\frac{3\sec\theta\tan\theta d\theta}{27\sec^2\theta\tan\theta}$$

$$=\frac{1}{9}\lim_{l\to3^+}\int_{l}^{c}\frac{1}{\sec\theta}d\theta+\frac{1}{9}\lim_{u\to\infty}\int_{c}^{u}\frac{1}{\sec\theta}d\theta=\frac{1}{9}[\lim_{l\to3^+}\int_{l}^{c}\cos\theta d\theta+\lim_{u\to\infty}\int_{c}^{u}\cos\theta d\theta]$$

$$=-\frac{1}{9}[\lim_{l\to3^+}\sin\theta\Big|_{l}^{c}+\lim_{u\to\infty}\sin\theta\Big|_{c}^{u}]=-\frac{1}{9}[\lim_{l\to3^+}\frac{\sqrt{x^2-9}}{x}\Big|_{l}^{c}+\lim_{u\to\infty}\frac{\sqrt{x^2-9}}{x}\Big|_{c}^{u}]$$

$$=-\frac{1}{9}[\frac{\sqrt{c^2-9}}{c}-\lim_{l\to3^+}\frac{\sqrt{l^2-9}}{l}+\lim_{u\to\infty}\frac{\sqrt{u^2-9}}{u}-\frac{\sqrt{c^2-9}}{c}]$$

$$=-\frac{1}{9}[\lim_{u\to\infty}\frac{\sqrt{u^2-9}}{u}-\lim_{l\to3^+}\frac{\sqrt{l^2-9}}{l}]=-\frac{1}{9}(1-0)=-\frac{1}{9}$$

8. Evaluate the improper integrals and determine their convergence.

$$a)\int_{-3}^{+\infty}\frac{1}{x^2+8x+17}dx \qquad b\int_{-\infty}^{+\infty}\frac{1}{x^2+6x+12}dx \qquad c)\int_{-\infty}^{+\infty}\frac{1}{4x^2+4x+17}dx$$

Solution: Use completing the square to rearrange for tangent substitution.

$$a)\int_{-3}^{+\infty}\frac{1}{x^2+8x+17}dx=\lim_{u\to+\infty}\int_{-3}^{u}\frac{1}{(x+4)^2+1}=\lim_{u\to+\infty}\tan^{-1}(x+4)\Big|_{-3}^{u}=\frac{\pi}{4}$$

$b)$ By completing the square, $x^2+6x+12=(x+3)^2+3$. So, using $u=x+3$,

$$\int_{-\infty}^{+\infty}\frac{dx}{x^2+6x+12}=\lim_{l\to-\infty}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{l}^{0}+\lim_{u\to\infty}\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\Bigg|_{0}^{u}=\frac{\pi}{\sqrt{3}}$$

$$c)\int_{-\infty}^{+\infty}\frac{dx}{4x^2+4x+17}=\lim_{l\to-\infty}\frac{1}{2}\int_{l}^{0}\frac{1}{u^2+16}du+\lim_{l\to\infty}\frac{1}{2}\int_{0}^{u}\frac{1}{u^2+16}du=\frac{\pi}{8}$$

9*. Show that

a) $f(x)=\int_0^x\sqrt{1+t^4}\,dt$ has an inverse and find $(f^{-1})'(c)$ where $c=f(1)$.

b)* $F(x)=\int_1^{x^3}\sqrt{3+t^4}\,dt$ has an inverse and find $(F^{-1})'(0)$

Solution:

a) $f(x)=\int_0^x\sqrt{1+t^4}\,dt\Rightarrow f'(x)=\sqrt{1+x^4}>0$ which means f is increasing.

Thus, it has an inverse. Besides, $c=f(1)\Rightarrow f^{-1}(c)=\frac{1}{f'(1)}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$.

b) $F(x)=\int_1^{x^3}\sqrt{3+t^4}\,dt\Rightarrow F'(x)=3x^2\sqrt{8+x^6}\geq 0$ which means f is increasing.

Thus, it has an inverse. Now find a such that $(F^{-1})'(c)=\frac{1}{F'(a)}$

That is $F(a)=0\Rightarrow\int_1^{a^3}\sqrt{3+a^4}\,dt=0\Rightarrow a^3=1\Rightarrow a=1$.

So, $F'(x)=3x^2\sqrt{3+x^6}\Rightarrow F'(1)=3\sqrt{4}=6\Rightarrow(F^{-1})'(0)=\frac{1}{F'(1)}=\frac{1}{6}$

Review Problems on Chapter-6

1. If $h(2)=3e^2$ and $h(0)=-5$, evaluate $\int_0^2 e^{-x}[h(x)-h'(x)]dx$. **Answer:** 8

2. Evaluate the following integrals using substitution.

$a)\int(2x^3+x)(x^4+x^2)^{49}dx \quad b)\int x^2\cos(x^3)dx \quad c)\int x^4e^{x^5}dx$

$d)\int(x+\frac{1}{x})^5(1-\frac{1}{x^2})dx \quad e)\int\frac{4x^3-2x}{\sqrt{x^4-x^2+4}}dx \quad f)\int\frac{6x^2+2\cos x}{x^3+\sin x}dx$

$g)\int(x^2+1)\sqrt{x^3+3x}dx \quad h)\int(1+2x)e^{x+x^2}dx \quad i)\int 3x^2\cos x^3e^{\sin x^3}dx$

Answer : $a)\ \frac{1}{50}(x^4+x^2)^{50}+c \quad b)\ \frac{1}{3}\sin(x^3)+c \quad c)\frac{1}{5}e^{x^5}+c$

$d)\ \frac{1}{6}(x+\frac{1}{x})^6+c \quad e)\ 2\sqrt{x^4-x^2+4}+c \quad f)\ 2\ln|x^3+\sin x|+c$

$g)\ \frac{2}{9}(x^3+3x)^{3/2}+c \quad h)\ e^{x+x^2}+c \quad i)\ e^{\sin x^3}+c$

3. Evaluate the following integrals using integration by parts

$a)\int\cos^{-1}(-7x)dx \quad b)\int 6x^3e^{x^2}dx \quad c)\int(1+x)e^xdx \quad d)\int\frac{\tan^{-1}x}{x^2}dx$

$e)\int\tan x\ln(\cos x)dx \quad f)\int 2x^3\cos x^2dx \quad g)\int\frac{1}{x^3}\cos\frac{1}{x}dx \quad h)\int\sin(\ln x)dx$

$i)\int\sin x\ln(\cos x)dx \quad j)\int e^{\sqrt{x}}dx \quad k)\int\sin\sqrt{x}dx \quad l)\int\sec^{-1}\sqrt{x}dx$

Answer : $a)\ x\cos^{-1}(-7x)+\frac{1}{7}\sqrt{1-49x^2}+c \quad b)\ 3x^2e^{x^2}-3e^{x^2}+c$

$c)\ xe^x+c \quad d)\ -\frac{\tan^{-1}x}{x^2}+\ln|x|-\frac{1}{2}\ln(x^2+1)+c \quad e)\ -\frac{1}{2}[\ln(\cos x)]^2+c$

$f)\ x^2\sin x^2+\cos x^2+c \quad g)\ -\frac{1}{x}\sin\frac{1}{x}-\cos\frac{1}{x}+c$

$h)\ \frac{1}{2}[x\sin(\ln x)-x\cos(\ln x)]+c \quad i)\ \cos x(1-\ln\cos x)+c$

$j)2e^{\sqrt{x}}(\sqrt{x}-1) \quad k)\ 2\sin\sqrt{x}-2\sqrt{x}\cos\sqrt{x} \quad l)\ x\sec^{-1}\sqrt{x}-\sqrt{x-1}+c$

4. Evaluate the following definite integrals

$a)\int x^7(x^4+2)^5dx \quad b)\int \sqrt{x}(\sqrt{x}-1)^7dx \quad c)\int \cos^3 x\ln x(\sin x)dx$

$d)\int \frac{2x-x^3}{(x^2+2)^3}dx \quad e)\int \frac{2x^3+6x}{x^4+6x^2+10}dx \quad f)\int \frac{2\cos(\sec^{-1}x^2)}{x\sqrt{x^4-1}}dx$

5. Suppose $f(x)=\begin{cases}\sin x, x\le 0\\ 3x^2\sqrt{x^3+1}, x>0\end{cases}$. Evaluate $\int_{-\pi}^{2} f(x)dx$. **Answer:**______

6. Evaluate the following definite integrals

$a)\int \frac{dx}{\sqrt{9-4x^2}} \quad b)\int \frac{dx}{x\sqrt{x^2-25}} \quad c)\int \frac{x^2}{\sqrt{25-x^2}}dx \quad d)\int \frac{dx}{x^4\sqrt{x^2+3}}$

$e)\int \frac{dx}{(3-x^2)^{3/2}} \quad f)\int \frac{dx}{\sqrt{4-2x^2}} \quad g)\int \frac{\sqrt{x^2-4}}{x}dx \quad h)\int \frac{\sqrt{x^2+4}}{x}dx$

$i)\int \frac{dx}{4+9x^2} \quad j)\int \frac{dx}{\sqrt{3-2x^2}} \quad k)\int \frac{dx}{x\sqrt{9x^2-16}} \quad l)\int \frac{x^3}{\sqrt{1-x^2}}dx$

$m)\int \frac{\sqrt{x^4-1}}{x}dx \quad n)\int \frac{\sqrt{1-x^2}}{x^4}dx \quad o)\int \frac{x^3}{\sqrt{9-x^2}}dx \quad p)\int \frac{x^3}{\sqrt{1+x^2}}dx$

Answer : $a)\frac{1}{2}\sin^{-1}(\tfrac{2}{3}x)+c \quad b)\frac{1}{5}\sec^{-1}(\tfrac{1}{5}x) \quad c)\frac{25}{2}\sin^{-1}\frac{x}{5}-\frac{x\sqrt{25-x^2}}{2}+c$

$d)\ \frac{1}{9}\left(\frac{\sqrt{x^2+3}}{x}-\frac{(x^2+3)^{3/2}}{3x^3}\right) \quad e)\ \frac{x}{3\sqrt{3-x^2}} \quad f)-\frac{1}{2}\ln\left|\frac{2}{\sqrt{2}x}+\frac{\sqrt{2-2x^2}}{\sqrt{2}x}\right|$

$g)\ \sqrt{x^2-4}-2\sec^{-1}\frac{x}{2}+c \quad h)\ \sqrt{x^2+4}+2\ln\left|\frac{\sqrt{x^2+4}-2}{x}\right|+c$

$i)\ \frac{1}{6}\tan^{-1}\left(\frac{3x}{2}\right)+c \quad j)\ \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{\sqrt{2}x}{\sqrt{3}}\right)+c \quad k)\ \frac{1}{4}\sec^{-1}\left(\frac{3x}{4}\right)+c$

$l)\ (1-x^2)^{3/2}-\sqrt{1-x^2}+c \quad m)\ \tfrac{1}{2}(\sqrt{x^4-1}-\sec^{-1}x^2)+c \quad n)\ \frac{(1-x^2)^{3/2}}{3x^3}$

$o)\ -\tfrac{1}{3}\sqrt{9-x^2}(x^2+18)+c \quad p)\ \tfrac{1}{3}\sqrt{1+x^2}(x^2-2)+c$

7. Evaluate using trigonometric substitutions.

a) $\int \frac{x^3}{9+x^8}dx$ b) $\int \frac{dx}{x^2\sqrt{x^2-1}}$ c) $\int \frac{dx}{(4x^2-9)^{3/2}}$ d*) $\int \frac{\sqrt{x^2+16}}{x^4}dx$

e) $\int \frac{e^x}{\sqrt{1-e^{2x}}}dx$ f) $\int \frac{e^{-x}}{\sqrt{1-e^{-2x}}}dx$ g) $\int \frac{dx}{\sqrt{1+e^{-2x}}}$

Answer : a) $\frac{1}{12}\tan^{-1}(\frac{x^4}{3})+c$ b) $\frac{\sqrt{x^2-1}}{x}+c$ c) $-\frac{x}{9\sqrt{4x^2-9}}+c$

d) $-\frac{(x^2+16)^{3/2}}{48x^3}$ e) $\sin^{-1}(e^x)$ f) $-\tan^{-1}(e^{-x})$ g) $\ln(\sqrt{1+e^{2x}}+e^x)$

8*. Given $F(x)=\int_1^{x^3}\sqrt{1+3t^4}\,dt$. Find the equation of a tangent line at $x_0=1$.

9. Suppose f' is continuous function on the interval $[1,4]$ with $f(1)=-2$ and $f(4)=19$. Then, evaluate $\int_1^4 \frac{f'(x)}{\sqrt{6+f(x)}}dx$. **Answer:** 6

10. Suppose the curve $y=f(x)$ passes through the origin and the point $(1,2)$. Then, the value of the integral $\int_0^1[f'(t)+3\pi\sin \pi t]dt$ is _____. **Answer:** 8

11*. Given $G(x)=6+\int_{x^3}^{8}\sqrt{1+\frac{1}{64}t^3}\,dt$. Find $G'(x)$ and write the equation of the tangent line to the graph of $G(x)$ at a point where $x=G^{-1}(6)$.

Answer : $G'(x)=-3x^2\sqrt{1+\frac{1}{64}x^9}$; $l: 36x+y=72$

12. Evaluate using trigonometric substitutions.

a) $\int \frac{x+1}{\sqrt{8+2x-x^2}}dx$ b) $\int \frac{1}{x^2+4x+8}dx$ c) $\int \frac{1}{(x^2-2x-3)^{\frac{3}{2}}}dx$

Answer : a) $-\sqrt{8+2x-x^2}+2\sin^{-1}(\frac{x-1}{3})$ b) $\frac{1}{2}\tan^{-1}(\frac{x+2}{2})$ c) $\frac{1-x}{4\sqrt{x^2-2x-3}}$

13*. Determine whether the improper integrals are convergent or divergent.

$a)\int_3^{\infty}\frac{1}{x^2\sqrt{x^2-9}}dx$ $\quad b)\int_1^{\infty}x^2e^{-x^3}dx$ $\quad c)\int_0^{\infty}\frac{1}{\sqrt{x}(x-1)}dx$

$d)\int_3^{+\infty}\frac{\sqrt{x^2-9}}{x}dx$ $\quad e)\int_0^{\infty}\frac{e^{2x}}{e^{4x}+9}dx$ $\quad$ **Answer** : $a)\ -\frac{1}{9}$

14. Find a and b so that $\int_a^b (x^2-16)dx$ is minimum. **Answer** : $a=-4, b=4$

15*. Evaluate the integral $\int_1^{e^{\pi}}\cos(\ln x)dx$ using appropriate technique.

Answer: $-\frac{1}{2}(e^{\pi}+1)$

16*. Given $G(x)=6+\int_{x^3}^{8}\sqrt{1+\frac{1}{64}t^3}\,dt$. Find $G'(x)$ and write the equation of the tangent line to the graph of $G(x)$ at a point where $x=G^{-1}(6)$.

Answer : $G'(x)=-3x^2\sqrt{1+\frac{1}{64}x^9}; l:36x+y=72$

17. For what values of **p** does the improper integral $\int_2^{\infty}\frac{1}{x(\ln x)^{p/5}}$ converge?

Answer : $p>5$

18*. Given $f(x)=\int_2^{x}\sqrt{1+t^2}\,dt$. Find $(f^{-1})'(0)$. **Answer** : $\sqrt{5}/5$

19. Find the area of a region bounded by the graphs of $f(x)=8x$ and $f(x)=2x^3$

20. Find the value of c for which the improper integral $\int_0^{\infty}\left(\frac{1}{\sqrt{x^2+1}}-\frac{c}{x+1}\right)dx$ converges and evaluate the integral for this value of c. **Answer** : $c=1;\ \ln 2$

21. Show that $a)\int_{-\infty}^{\infty}\frac{1+x}{1+x^2}dx=\pi$ $\quad b)\int_0^{\infty}\frac{x}{(x^2+4)^2}dx=\frac{1}{32}$